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Dynamic response of a gyrocompass

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DYNAMIC RESPONSE OF A GYROCOMPASS

BY

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A

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ABSTRACT

A gyrocompass with three degrees of freedom is studied and an attempt made to solve the exact equations of motion numerically. The equations are obtained by using the Lagrange formulation of the problem. The kinetic energy and the potential energy of the system are determined from the energies of the various components. These are then substituted into Lagrange's equation to obtain the three equations of motion for the three angular coordinates. As few restrictive assumptions as possible are made during this development. From these exact equations of motion an approximate analytical solution is obtained by making several assumptions and then solving the linearised equations. As the requirement of a gyrocompass is to point North at any instant, an equilibrium configuration of the system is sought to find the inclination of the gyro axis with the horizontal in a position of steady motion.

The exact equations of motion are complicated. Hence they are solved with the help of a high speed digital computer using numerical methods for solving differential equations. The methods used are the Runge-Kutta method of order 4, and the Hamming method of order 1. A comparison is made between the two methods to find which is the better for solving such a system of equations. A comparison is also made between the numerical solution to the exact equations of motion and the approximate analytical solution to check the effect of the approximations made. Graphs are plotted from the results obtained. Suggestions are made for further work.

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TABLE OF CONTENTS

LIST OF SYMBOLS

(In order of appearance)

w = Weight of the pendulous weight

m = Mass of the pendulous weight

X-Y-Z = Set of axes

1-2-3 = Set of axes

4-5-6 = Set of axes

7-8-9 = Set of axes

$\hat{i}, \hat{j}, \hat{k}, \hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4, \hat{e}_5, \hat{e}_6, \hat{e}_7, \hat{e}_7^*, \hat{e}_8, \hat{e}_9$ = Respective unit vectors

ψ = Angular displacement in precession

α = Angular displacement in nutation

ϕ = Angular displacement of the gyrorotor

$\overline{\omega}$ = Angular velocity in general

Ω = Magnitude of the angular velocity of the earth about its North-South axis

λ = Latitude in radians

Ω_c = $\Omega \cdot \cos\lambda$

Ω_s = $\Omega \cdot \sin\lambda$

T = Kinetic energy

U = Potential energy

R = Radius of the earth

ℓ = Length of the pendulous weight (moment arm)

e = Eccentricity of the pendulous weight from the center

$\mathbf{r}$ = Position vector

$\overline{V}$ = Velocity vector

I = Mass moment of inertia

x_i = Generalized coordinate

$\dot{x}_i$ = Time derivative of x_i

Q_i = Generalized force in the direction of x_i

L = Lagrangian function

R_i = Nonconservative forces

D_{12}, D_{34}, D_{56} = Damping coefficients

α_o = Value of α in the equilibrium position

LIST OF FIGURES

I. INTRODUCTION

A. GENERAL

"Developments in the exploration of space have brought to the forefront many new problems in science and technology. Motion in outer space poses unusual dynamic problems the solution of which requires a thorough knowledge and understanding of the pertinent dynamical principles and techniques of analysis. Three-dimensional attitude problems involving gyroscopic phenomena play an important part in the behavior of guidance instrumentation and in establishing the motion of space vehicles. For most problems, the high speeds encountered in space motion are not sufficiently great, in a relativistic sense, to invalidate Newton's laws of motion, and a knowledge of classical mechanics serves as adequate foundation for a description of the phenomena encountered" - W. T. Thompson.

The gyrocompass is the subject considered in the present study. This instrument is often employed for inertial navigation. In its simplest form, it consists of a rotor, an inner gimbal to which it is attached, an outer gimbal enclosing the framework, and a pendulous weight, attached to the inner gimbal. The gyrocompass is an ingenious application of the gyroscope. It is affected by gravity and also by the earth's rotation, so that the gyro axis is in equilibrium only when it points North i.e., when it lies in the plane formed by the local vertical and the earth's North-South axis. If the compass is disturbed so that it points away from the North, the moments acting on it will restore it to the correct North position. If the axis of the gyro is vertical, it can be used in aviation to locate the vertical. Due to the extremely high angular velocity of the rotor, the analysis of the motion of the gyrocompass is generally difficult. Attempts have been made to obtain a simplified solution to the problem and it has been found that the energy method presents the simplest means of obtaining the differential equation of motion. In the present discussion, the Lagrangian approach is employed throughout to obtain the differential equations of motion. Due to a simultaneous motion in all the three angular directions, the equations of motion are non-linear and coupled. It is of principal concern to solve for the motion in all angular directions. The complicated nature of the equations gives rise to difficulties in solving them. Hence, assumptions are made to linearize the equations.

An effort is also made to utilize the numerical methods available to solve the equations on a digital computer. The Runge-Kutta method and the Hamming's method of orders four and one respectively, are found to be the most suitable for the form of equations under discussion. This approach is different from the methods which have been applied previously. It also is the purpose of this research to study the accuracy with which the solutions are obtained with the help of a computer. For this purpose, the problem is solved both analytically (after making several assumptions) and on a digital computer.

The comparison between the usefulness of the two methods and a discussion of their applications forms the basis of this research.

However, it should be noted that although a gyrocompass has been used in this analysis, the analysis can as well be applied to other similar instruments.

B. REVIEW OF LITERATURE

The analysis of the motion of a gyrocompass is generally covered in works on aerospace mechanics. However, in every case, several assumptions are made to simplify the nature of the equations.

W. T. Thompson[1] has made the following assumptions: (i) The angular deviation of the spin axis out of the meridian plane, ψ, and its inclination above the horizontal, α, are small. (ii) The angular momenta about the precession as well as the nutation axes are negligible in comparison with that about the spin axis. His conclusions are based on these assumptions.

Because of the wide applications of the gyrocompass, several technical papers on the subject have appeared recently.

M. Kayton[10] describes the drift of the stable platform caused by the gyrorotor unbalance.

D. R. Merkin[11] has shown that the previously established stability conditions for precessional motion of the frame can be extended to a general case.

W. G. Wing[12] explains the uses of a gyrostabilised platform and shows that inertial systems, based on Newton's laws of motion, can measure the total acceleration vector of a vehicle and calculate vehicle speed and position.

V. N. Koshlykov[13] reduces the equations of motion of the gyro-horizon-compass to a system with constant coefficients and presents a rigorous analytic justification of passage to the simplified equations of J. M. Geckeler.

A. M. Tabarovskii[14] has obtained the approximate equations of motion and has then reduced the integration of these equations to quadratures. The method of Chetaev has been used in the discussion on stability.

A. G. Lindgreen and M. P. Feder[15] have discussed the stability and control of gyroscopic bodies with two degrees of freedom.

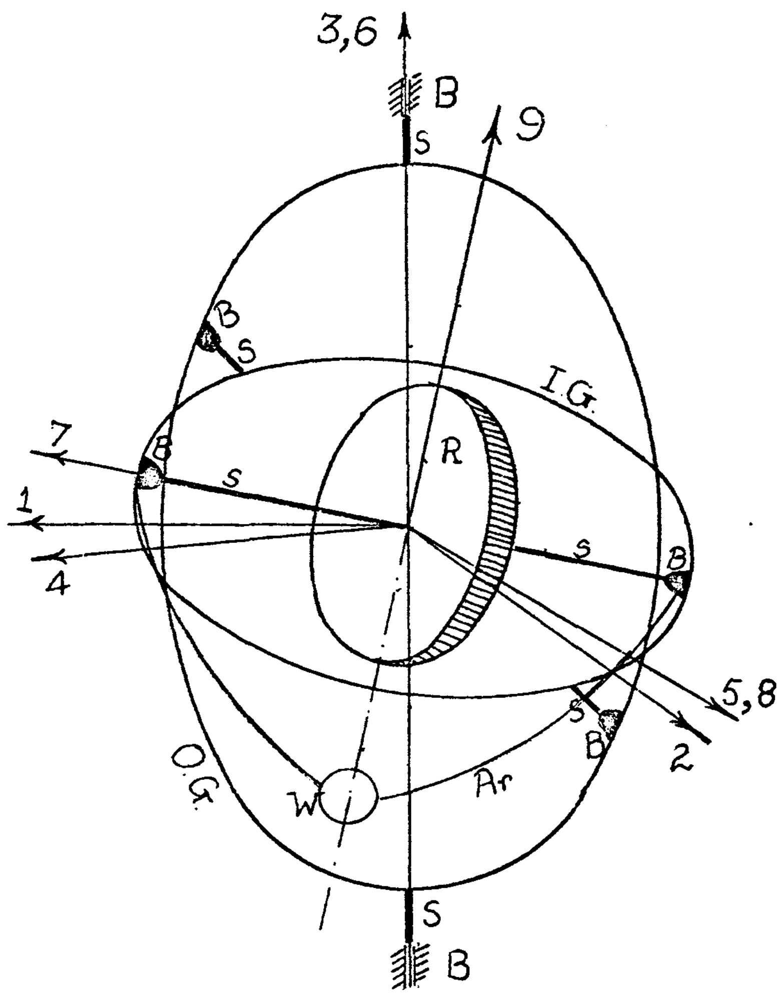


Figure 1.

MECHANICAL ARRANGEMENT OF THE GYROCOMPASS

R = Rotor

I.G. = Inner Gimbal

O.G. = Outer Gimbal

w = Pendulous Weight (point mass)

s = Shaft

B = Bearing

A_r = Massless arm to which the pendulous weight is attached

II. DEVELOPMENT OF EQUATIONS OF MOTION

A. MECHANICAL ARRANGEMENT

The gyrocompass was found to be applicable to marine navigation in the middle of the eighteenth century. Until then, the magnetic compass was used in all navigational instruments. The purpose of overcoming the defects of the magnetic compass was indeed served; although, it was only around 1910 that really successful instruments were developed. There have been several improvements through the years. However, the theory behind its working has always been, and shall always be the same. The modern gyrocompass essentially consists of a rotor (the gyroscope) whose motion is governed by the combined action of the earth's rotation and the moment produced by a gravitational force. A small mass is generally arranged to exert a gravitational torque about the 8-axis (see Figure 1). The rotor-axis 7 is always expected to point to the North. If the axis is deflected from the North, it will execute a slow oscillation and ultimately come to rest in the direction pointing towards the North. The mechanical arrangement of the compass, in its simplest form is such that the gyro-rotor is mounted on a shaft, which is connected to bearings rigidly attached to the inner of the two supporting rings, called the "gimbals". The inner gimbal is a ring with a built-in or a rigidly fixed arm carrying the pendulous weight. The arm is responsible for always keeping the center of mass of the pendulous weight in the -9 direction. However, it is necessary that no relative motion between the pendulous weight and the inner gimbal is permitted, as the only purpose of the pendulous weight is to provide a restoring torque about the 8-axis, and thus help in bringing the nutational motion of the gyro to rest. Again, the inner gimbal is itself mounted on another shaft which connects it to the outer supporting gimbal by means of a bearing which is similar to the bearing connecting the gyro-shaft to the inner gimbal. The inner gimbal is free to rotate about its shaft. The outer gimbal is the supporting ring that encloses the entire arrangement, and it is mounted on a vertical shaft about which it rotates. There are several ways of externally mounting the entire instrument but as they do not contribute to the motion of the components, they are not considered in the present discussion.

In the modern gyrocompass, different means of exerting torque, than the pendulous weight are used. An example of this is the Sperry compass which uses a Servo-system instead of the pendulous weight. The gyro-axis is generally the pointer which indicates the direction. The above-mentioned are only the essential features of the gyrocompass. Many other accessories are also provided to improve the performance of the instrument.

B. THE REFERENCE COORDINATE SYSTEM

In developing the equations of motion for any body, it is necessary to specify the reference coordinate systems. Newtonian mechanics has no single, preferred frame of reference. The Inertial or Newtonian reference frame is defined as any rigid set of coordinate axes for which particle motion relative to the axes is described by Newton's laws of motion.

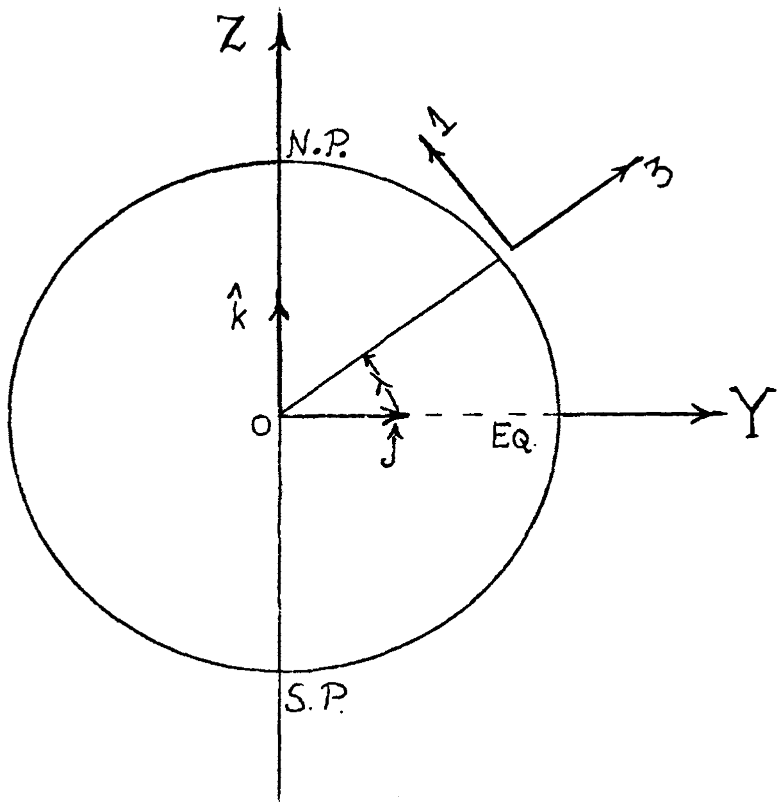


Figure 2.

REFERENCE COORDINATE SYSTEMS X-Y-Z AND 1-2-3

N.P. = North Pole

S.P. = South Pole

EQ = Equator

For the present problem, the primary reference system will be the XYZ system with its origin located at the center of the earth (see Figure 2). An auxiliary reference system 123 is attached to the earth, on its surface, at latitude λ. The Z axis always passes through the North pole while the X and the Y-axes always lie in the equatorial plane. The 123-axes are situated so that they always remain fixed at the same latitude and longitude, i.e., there is no relative motion between the 123-axes, and the surface of the earth. The 3 axis always points in the direction of the local vertical and the 1 axis always points North in the direction of the local horizontal. Hence the 2 axis always points to the West in the direction of the local horizontal. The earth's attractive force acts in the $-\hat{e}_3$ direction.

The origin of the 123-axes is the same as the center of mass of the gyrocompass rotor. A second auxiliary system 456 (see Figure 1) is attached to the outer gimbal, such that axis-6 always points to the local vertical. Yet another auxiliary system 789 is attached to the inner gimbal, in such a manner that the spin axis of the rotor is the 7 axis. The origins of the 456 and the 789 systems are coincident with that of the 123 system, as can be easily seen from Fig. 1.

C. ASSUMPTIONS

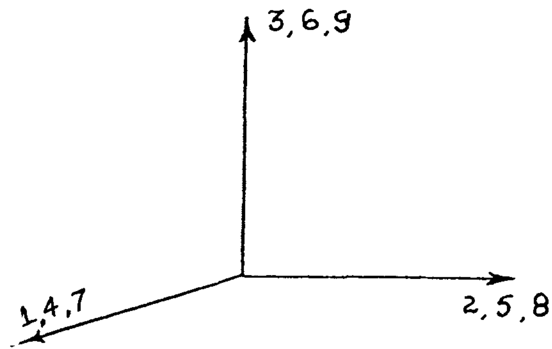
In this thesis, although an attempt is made to obtain the equations of motion for as general a case as possible, it was found that some assumptions were necessary in order to reduce the complexity of the equations of motion. However, care has been taken to see that the number of assumptions made is as small as possible, and that these assumptions do not affect the results signigicantly. The assumptions are listed as follows:

1. The primary reference frame for any rotational motion should be a Newtonian reference frame. Although, no single preferred frame of reference can be labelled as the Newtonian reference frame, one might consider a system at the center of the sun or at the center of mass of the solar system, and nonrotating with respect to the so-called fixed stars as the most adequate reference frame. But, in a case like the one under discussion, where distances traveled are short relative to the earth's radius, a system fixed at the center of the earth and translating with it would not be an unreasonable approximation to a Newtonian reference system. Hence the XYZ system is considered as a Newtonian reference frame throughout the discussion.

2. The earth is assumed to be rotating about the Z-axis at a constant angular velocity of Ω radians per second, where

$$\Omega = 7.27 \times 10^{-5} \text{ rad/sec}$$

3. The 123 axes which are attached to the center of mass of the gyrocompass are assumed to remain at a constant latitude, as well as at a constant longitude. In other words, the gyrocompass, or the vehicle carrying it, is assumed to remain stationary. However, this assumption does not affect the results significantly.

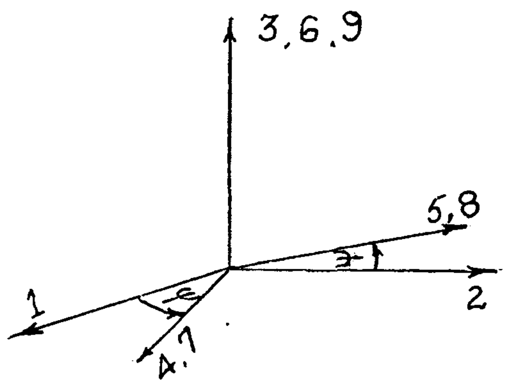


Aligned

$\psi = 0$

$\alpha = 0$

Figure 3.

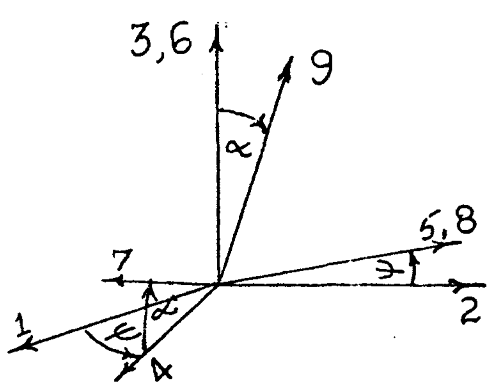


1^{st} Rotation, ψ about 3-axis

$\psi = \psi$

$\alpha = 0$

Figure 4.



2^{nd} Rotation, α about 8-axis

$\psi = \psi$

$\alpha = \alpha$

Figure 5.

UNIT VECTOR TRANSFORMATION

4. Actually, the gyrocompass will be situated at some height above the ground level, i.e. the center of mass of the gyro or the origin of the 123 system will remain at a certain height from the earth's surface. But this height being extremely small in comparison to the radius of the earth, is neglected and the distance between the center of the earth and the origin of the 123 system is considered to be the same as the radius of the earth.

D. UNIT VECTOR TRANSFORMATION EQUATIONS

The general motion of the gyrocompass is such that the rotor rotates about the 7 axis with a velocity of $\dot{\phi}$ radians per second. The outer gimbal is free to rotate about the 6-axis, and the angular displacement in this direction is ψ radians. This motion is known as "precession". The inner gimbal is free to rotate about the 8 axis, and the angular displacement in this direction is α radians. This motion is termed as "nutation". All the rotations are either clockwise or counter-clockwise, according to the right-hand screw rule.

The origin of the 789 system is coincident with that of the 123 and the 456 systems; the rotation of the 456 system with respect to the 123 system is specified by the angle ψ, while rotating the 456 axes about the 3 axis (and the 6 axis) through an angle ψ. The rotation of the 789 system relative to the 123 system is specified by the angles ψ and α. The rotation through the angle ψ corresponds to rotating 789 away from alignment with the 123 system through ψ radians, about the 3 (and the 6, and the 9) axis. Then the 789 system is rotated through

α about the 8 axis. It is important to preserve the order of these rotations as finite rotations do not add vectorially. The respective rotations can be visualized from figures 3, 4, and 5.

Now that a complete picture of the general motion is in mind, it will be convenient to present the unit vector transformation equations.

Let a unit vector along the 1, 2 and 3 axis be $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ respectively.

Let $\hat{e}_4$, $\hat{e}_5$, $\hat{e}_6$ and $\hat{e}_7$, $\hat{e}_8$, $\hat{e}_9$ be similar notations.

The relationship between 456 and 123, from Figure 5 can be written as

$$\begin{aligned}
\hat{e}_4 &= \cos\psi\cdot\hat{e}_1 + \sin\psi\cdot\hat{e}_2 + 0\cdot\hat{e}_3 \\
\hat{e}_5 &= -\sin\psi\cdot\hat{e}_1 + \cos\psi\cdot\hat{e}_2 + 0\cdot\hat{e}_3 \\
\hat{e}_6 &= 0\cdot\hat{e}_1 + 0\cdot\hat{e}_2 + 1\cdot\hat{e}_3
\end{aligned}$$

Or, representing in matrix notation,

$$\begin{bmatrix} \hat{e}_4 \\ \hat{e}_5 \\ \hat{e}_6 \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} \tag{1}$$

Similarly, 789 and 456 are related by,

$$\begin{aligned}
\hat{e}_7 &= \cos\alpha\cdot\hat{e}_4 + 0\cdot\hat{e}_5 + \sin\alpha\cdot\hat{e}_6 \\
\hat{e}_8 &= 0\cdot\hat{e}_4 + 1\cdot\hat{e}_5 + 0\cdot\hat{e}_6 \\
\hat{e}_9 &= -\sin\alpha\ \hat{e}_4 + 0\cdot\hat{e}_5 + \cos\alpha\cdot\hat{e}_6
\end{aligned}$$

In matrix notation,

$$\begin{bmatrix} \hat{e}_7 \\ \hat{e}_8 \\ \hat{e}_9 \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \hat{e}_4 \\ \hat{e}_5 \\ \hat{e}_6 \end{bmatrix} \tag{2}$$

Substituting equation (1) into equation (2), the relationship between 789 and 123 is

$$\begin{bmatrix} \hat{e}_7 \\ \hat{e}_8 \\ \hat{e}_9 \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & o & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ e_3 \end{bmatrix}$$

$$\begin{bmatrix} \hat{e}_7 \\ \hat{e}_8 \\ \hat{e}_9 \end{bmatrix} = \begin{bmatrix} \cos\psi\cdot\cos\alpha & \cos\alpha\cdot\sin\psi & \sin\alpha \\ -\sin\psi & \cos\psi & 0 \\ -\sin\alpha\cdot\cos\psi & -\sin\alpha\cdot\sin\psi & \cos\alpha \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} \tag{3}$$

From equations (3), an inverse relationship between 123 and 789 can also be obtained as

$$\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \begin{bmatrix} \cos\psi\cdot\cos\alpha & -\sin\psi & -\cos\psi\cdot\sin\alpha \\ \sin\psi\cdot\cos\alpha & \cos\psi & -\sin\psi\cdot\sin\alpha \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \hat{e}_7 \\ \hat{e}_8 \\ \hat{e}_9 \end{bmatrix} \tag{4}$$

This sums up all the unit vector transformation equations.

E. ANGULAR VELOCITY VECTORS

With all the unit vector transformation equations at hand, the determination of the angular velocities of each component of the gyrocompass is simplified. In order to obtain the angular velocity vectors of the rotor with respect to the XYZ system, we shall proceed step by step, first determining the velocity of 123 with respect to XYZ, then of 456 with respect to 123, then of 789 with respect to 456, and finally of the rotor with respect to 789. The intermediate step of determining the velocities of 456 may be omitted as a relationship between 789 and 123 has already been obtained.

The angular velocity of the earth, with respect to the XYZ system is, $\bar{\omega}_{earth/XYZ} = \Omega\hat{k}$ where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along the X, Y and Z axis, respectively.

From Figure 2, $\Omega\hat{k} = \Omega\cos\lambda\hat{e}_1 + \Omega\sin\lambda\hat{e}_3$. Denoting $\Omega\cos\lambda$ by Ω_c, and $\Omega\sin\lambda$ by Ω_s, $\bar{\omega}_{earth/XYZ} = \Omega_c\hat{e}_1 + \Omega_s\hat{e}_3$.

There is no relative motion between 123 and the earth.

$$\bar{\omega}_{123/XYZ} = \bar{\omega}_{123/earth} + \bar{\omega}_{earth/XYZ}$$

$$\bar{\omega}_{123/XYZ} = \Omega_c\hat{e}_1 + \Omega_s\hat{e}_3$$

If $\bar{\omega}_{123/XYZ}$ is written as

$$\bar{\omega}_{123/XYZ} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3$$

we have

$$\omega_1 = \Omega_c$$

$$\omega_2 = 0$$

$$\omega_3 = \Omega_s$$

Outer Gimbal:

The angular velocity of the outer gimbal is given by

$$\bar{\omega}_{456/XYZ} = \bar{\omega}_{456/123} + \bar{\omega}_{123/XYZ}$$
$$= \dot{\psi}\hat{e}_3 + \bar{\omega}_{123/XYZ}$$
$$\bar{\omega}_{456/XYZ} = \Omega_c\hat{e}_1 + (\dot{\psi}+\Omega_s)\ \hat{e}_3$$
$$= \Omega_c(\cos\psi\hat{e}_4 - \sin\psi\hat{e}_5) + (\dot{\psi}+\Omega_s)\ \hat{e}_6$$

If $\bar{\omega}_{456/XYZ}$ is written as

$$\bar{\omega}_{456/XYZ} = \omega_4\hat{e}_4 + \omega_5\hat{e}_5 + \omega_6\hat{e}_6, \text{ then}$$
$$\omega_4 = \Omega_c\cos\psi$$
$$\omega_5 = -\Omega_c\sin\psi$$
$$\omega_6 = \dot{\psi} + \Omega_s$$

Inner Gimbal:

The angular velocity of the inner gimbal is given by

$$\bar{\omega}^{*(1)}_{789/XYZ} = \bar{\omega}_{789/123} + \bar{\omega}_{123/XYZ}$$
$$= -\dot{\alpha}\hat{e}_8 + \dot{\psi}\hat{e}_3 + \bar{\omega}_{123/XYZ}$$
$$= -\dot{\alpha}\hat{e}_8 + \Omega_c\hat{e}_1 + (\dot{\psi}+\Omega_s)\ \hat{e}_3$$

From equation (4),

$$\hat{e}_1 = \cos\psi \cdot \cos\alpha\hat{e}_7 - \sin\psi\hat{e}_8 - \cos\psi\sin\alpha\hat{e}_9$$
$$\hat{e}_3 = \sin\alpha\hat{e}_7 + 0\ \hat{e}_8 + \cos\alpha\hat{e}_9$$

(1) As the angular velocity of the inner gimbal, and that of the rotor, both correspond to the angular velocity of the 7-8-9 coordinate system with the slight difference of the rotor velocity term $\dot{\phi}$ in the angular velocity of the rotor, ω^*_{789} is used to denote the velocity of the inner gimbal, and $\bar{\omega}_{789}$ that of the rotor.

Substituting these values in the equation above,

$$\bar{\omega}^*_{789/XYZ} = -\dot{\alpha}\hat{e}_8 + \Omega_c[\cos\psi\cdot\cos\alpha\hat{e}_7 - \sin\psi\hat{e}_8$$
$$-\cos\psi\cdot\sin\alpha\hat{e}_9] +$$
$$(\dot{\psi}+\Omega_s)[\sin\alpha\hat{e}_7 + 0\ \hat{e}_8 + \cos\alpha\hat{e}_9]$$

If $\bar{\omega}^*_{789/XYZ} = \omega^*_7\hat{e}_7 + \omega_8\hat{e}_8 + \omega_9\hat{e}_9$, then

$$\omega^*_7 = \Omega_c\cos\psi\cdot\cos\alpha + (\dot{\psi}+\Omega_s)\sin\alpha$$
$$\omega_8 = -(\dot{\alpha}+\Omega_c\sin\psi)$$
$$\omega_9 = (\dot{\psi}+\Omega_s)\cdot\cos\alpha - \Omega_c\cdot\cos\psi\cdot\sin\alpha$$

Rotor:

The angular velocity of the rotor is given by,

$$\bar{\omega}_{rotor/XYZ} = \bar{\omega}_{rotor/\overset{*}{7}89} + \bar{\omega}^*_{789/XYZ}$$
$$= \dot{\phi}\hat{e}_7 + \bar{\omega}^*_{789/XYZ}$$
$$= \bar{\omega}_{789/XYZ}$$

If $\bar{\omega}_{789/XYZ} = \omega_7\hat{e}_7 + \omega_8\hat{e}_8 + \omega_9\hat{e}_9$,

$$\omega_7 = \Omega_c\cdot\cos\psi\cdot\cos\alpha + (\dot{\psi}+\Omega_s)\cdot\sin\alpha + \dot{\phi}$$
$$\omega_8 = -(\dot{\alpha}+\Omega_c\cdot\sin\psi)$$
$$\omega_9 = (\dot{\psi}+\Omega_s)\cdot\cos\alpha - \Omega_c\cdot\cos\psi\cdot\sin\alpha$$

F. TOTAL SYSTEM KINETIC ENERGY

In order to apply Lagrange's equation to the system, at a later stage, to obtain the equations of motion, it is necessary to determine the total kinetic energy of the system.

The total kinetic energy consists of the kinetic energy due to translational motion as well as rotational motion.

$$T_{gyro} = T_{trans} + T_{rot}$$

The translational kinetic energy is given by,

$$\text{K. E.} = 1/2\ (\text{mass}) \times (\text{velocity of center of mass})^2$$

The rotational kinetic energy is given by

$$\text{K. E.} = 1/2\ (\text{inertia}) \times (\text{angular velocity of center of mass})^2$$

Hence, T_{trans} can be obtained from,

$$T_{trans} = 1/2\ m_{rotor} \cdot v_o^2 + 1/2\ m_{I.G.} \cdot v_o^2 + 1/2\ m_{O.G.} \cdot v_o^2 + 1/2\ mv^2$$

where, I.G. = Inner gimbal

O.G. = Outer gimbal

m = mass of pendulous weight

v_o = magnitude of the velocity of the gyrocompass

= magnitude of the velocity of earth at E (see Fig. 6)

= (radius of earth) x (angular velocity of earth)

= constant

v = velocity of pendulous weight

Here, it is necessary to note that the kinetic energy due to translational motion of the pendulous weight is not included in the kinetic energy of the inner gimbal i.e., although the pendulous weight is built-in with the inner gimbal, both are considered individually.

Also, as the Lagrangian equations involve only the derivatives of the kinetic energy, and the mass as well as the velocity of all components except those of the pendulous weight are constants, terms in the expression for T_{trans} contributed by these components may be dropped out at this stage.

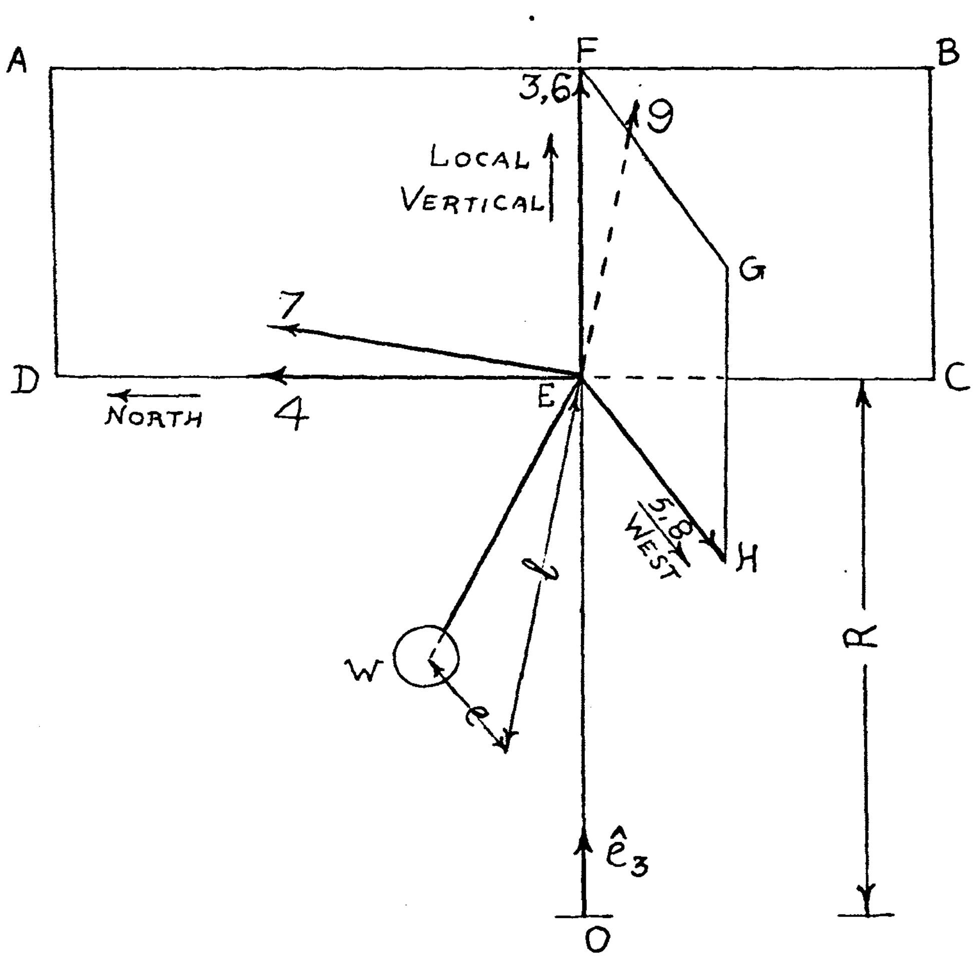


Figure 6.

POSITION OF PENDULOUS WEIGHT IN THE SYSTEM

ABCD = Plane formed by local vertical & North

EFGH = Plane formed by local vertical & West

O = Center of the earth

R = Radius of earth

ℓ = Length of pendulum

e = eccentricity in the direction of East

Hence, $$T_{trans} = 1/2\ mv^2$$

In order to calculate v, it is necessary to look at the position vector of the pendulous weight, relative to the origin of the primary reference system XYZ located at the center of the earth, and then differentiate it with respect to time. At this stage, in order to produce damping for the oscillation of the gyrocompass, let the pendulous weight be moved to the East of the centerline by a distance "e". This way, a moment about the 9 axis is introduced due to the eccentricity which provides damping in a general case.

From Figure 6, the position vector of the center of mass of the pendulous weight, $\bar{r}_{c.m.}$, with respect to 0 is given by

$$\bar{r}_{c.m.} = R\cdot\hat{e}_3 - \ell\cdot\hat{e}_9 - e\cdot\hat{e}_8$$

$$\bar{v} = \frac{d(\bar{r}_{c.m.})}{dt} = \dot{\bar{r}}_{c.m.} = R\cdot\dot{\hat{e}}_3 - e\cdot\dot{\hat{e}}_8 - \ell\cdot\dot{\hat{e}}_9$$

But, from elementary vector analysis, with the usual notations

$$\dot{\hat{e}}_3 = \bar{\omega} \times \hat{e}_3$$

Substituting,

$$\bar{v} = R[\bar{\omega}_{123} \times \hat{e}_3] - e[\bar{\omega}_{789} \times \hat{e}_8] - \ell[\bar{\omega}_{789} \times \hat{e}_9]$$

$$= R\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ 0 & 0 & 1 \end{vmatrix} - e\begin{vmatrix} \hat{e}_7 & \hat{e}_8 & \hat{e}_9 \\ \omega_7^* & \omega_8 & \omega_9 \\ 0 & 1 & 0 \end{vmatrix} - \ell\begin{vmatrix} \hat{e}_7 & \hat{e}_8 & \hat{e}_9 \\ \omega_7^* & \omega_8 & \omega_9 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= R(\omega_2\cdot\hat{e}_1 - \omega_1\cdot\hat{e}_2) - e(\omega_7^*\cdot\hat{e}_9 - \omega_9\cdot\hat{e}_7) - \ell(\omega_8\cdot\hat{e}_7 - \omega_7^*\cdot\hat{e}_8)$$

$$= R\omega_2\hat{e}_1 - R\omega_1\hat{e}_2 + (e\omega_9 - \ell\omega_8)\hat{e}_7 + \ell\omega_7^*\hat{e}_8 - e\omega_7^*\hat{e}_9$$

$$= -R\Omega_c[\sin\psi\cos\alpha\cdot\hat{e}_7 + \cos\psi\cdot\hat{e}_8 - \sin\psi\sin\alpha\cdot\hat{e}_9]$$

$$+ (e\omega_9 - \ell\omega_8)\hat{e}_7 + \ell\omega_7^*\cdot\hat{e}_8 - e\omega_7^*\hat{e}_9,$$

as $\omega_1 = \Omega_c$, $\omega_2 = 0$, and $\hat{e}_2 = [\sin\psi\cos\alpha\hat{e}_7 + \cos\psi\cdot\hat{e}_8 - \sin\psi\sin\alpha\cdot\hat{e}_9]$ from sections E and D, respectively.

Rearranging,

$$\overline{V} = (e\omega_9 - \ell\omega_8 - R\Omega_c\sin\psi\cos\alpha)\cdot\hat{e}_7 +$$

$$(\ell\omega_7^* - R\Omega_c\cos\psi)\cdot\hat{e}_8 + (R\Omega_c\sin\psi\sin\alpha - e\omega_7^*)\hat{e}_9$$

Substituting in equation for T_{trans},

$$T_{trans} = 1/2\ m\ [(e\omega_9 - \ell\omega_8 - R\Omega_c\sin\psi\cos\alpha)^2 +$$

$$(\ell\omega_7^* - R\Omega_c\cos\psi)^2 + (R\Omega_c\sin\psi\sin\alpha - e\omega_7^*)^2]$$

The kinetic energy due to rotational motion is given by,

$$T_{rot} = (T_{rot})_{O.G.} + (T_{rot})_{I.G.} + (T_{rot})_{rotor}$$

Define the inertias of various components as follows:

For O.G.,

$$I_{44} = C_o$$

$$I_{55} = I_{66} = A_o, \text{ due to symmetry}$$

For I.G.,

$$I_{77} = C_I$$

$$I_{88} = A_I$$

$$I_{99} = B_I$$

For rotor,

$$I_{77} = C$$

$$I_{88} = I_{99} = A, \text{ due to symmetry}$$

The rotational kinetic energy for each component is given by,

$$(T_{rot})_{O.G.} = 1/2[I_{44}\omega_4{}^2 + I_{55}\omega_5{}^2 + I_{66}\omega_6{}^2]$$
$$= 1/2[C_o\omega_4{}^2 + A_o(\omega_5{}^2 + \omega_6{}^2)]$$

$$(T_{rot})_{I.G.} = 1/2[I_{77}\omega_7^{*2} + I_{88}\omega_8{}^2 + I_{99}\omega_9{}^2]$$
$$= 1/2[C_I\omega_7^{*2} + A_I\omega_8{}^2 + B_I\omega_9{}^2]$$

$$(T_{rot})_{rotor} = 1/2[I_{77}\omega_7{}^2 + I_{88}\omega_8{}^2 + I_{99}\omega_9{}^2]$$
$$= 1/2[C\omega_7{}^2 + A(\omega_8{}^2 + \omega_9{}^2)]$$

The pendulous weight has no rotational motion. Thus,

$$T_{rot} = 1/2[C_o\omega_4{}^2 + A_o(\omega_5{}^2 + \omega_6{}^2) + C_I\omega_7^{*2} + C\omega_7{}^2$$
$$+ (A+A_I)\cdot\omega_8{}^2 + (A+B_I)\omega_9{}^2]$$

and,

$$T = T_{rot} + T_{trans}$$

gives

$$T = 1/2[C_o\omega_4{}^2 + A_o(\omega_5{}^2 + \omega_6{}^2) + (A+A_I)\omega_8{}^2 +$$
$$(A+B_I)\omega_9{}^2 + C\omega_7{}^2 + C_I\omega_7^{*2} + m\{(\ell\omega_7^* - R\Omega_c\cos\psi)^2$$
$$+ (R\Omega_c\sin\psi\sin\alpha - C\omega_7{}^*)^2 +$$
$$(e\omega_9 - \ell\omega_8 - R\Omega_c\sin\psi\cos\alpha)^2\}]$$

which is the expression for the total system kinetic energy.

G. EQUATIONS OF MOTION

The equations of motion of a system can be obtained by several different methods. One of the most convenient methods of obtaining the equations of motion is to apply Lagrange's equations to the system.

To use Lagrange's equation, it is necessary to define:

1. Generalized Coordinates: A set of independent coordinates that are necessary and sufficient to describe the motion of a system completely.

2. Holonomic System: A system described by n generalized coordinates $q_1, q_2, \ldots, q_n$ with m constraint equations of the form

$$\phi_i(q_1, q_2, \ldots, q_n, t) \leq 0 \qquad (i = 1, 2, \ldots m)$$

is a Holonomic system. In such a system, the required number of generalized coordinates is the same as the number of degrees of freedom.

3. Conservative System: A conservative system is one in which the work done is merely a function of position, and is independent of the path taken by the force doing the work. Thus, work done by a conservative force system around any closed path must be zero. Forces such as the frictional forces in the system are nonconservative.

Lagrange's equations for a holonomic system, such as ours can be written as

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i \qquad (5)$$

where

T = kinetic energy of the system

q_i; $i = 1,2,3$ = generalized coordinates ψ, α, and ϕ respectively

Q_i = generalized forces in the direction of the generalize coordinates

If the system is conservative, $Q_i = \frac{-\partial U}{\partial q_i}$,

where U = potential energy of the system.

Defining the Lagrangian function L as

$$L = T - U$$

the equations (5) now reduce to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

If the system is non-conservative,

$$Q_i = R_i - \frac{\partial U}{\partial q_i}, \text{ where}$$

R_i are the nonconservative generalized forces acting on the system Equations (5) now take the form,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = R_i ,$$ which is the general form of Lagrange's equations.

Returning to our system, let the damping forces due to viscous friction at the bearings, etc. be,

$R_1 = -D_{12}\dot{\psi}$ = torque damping the angular motion in the ψ direction

$R_2 = -D_{34}\dot{\alpha}$ = torque damping the angular motion in the α direction

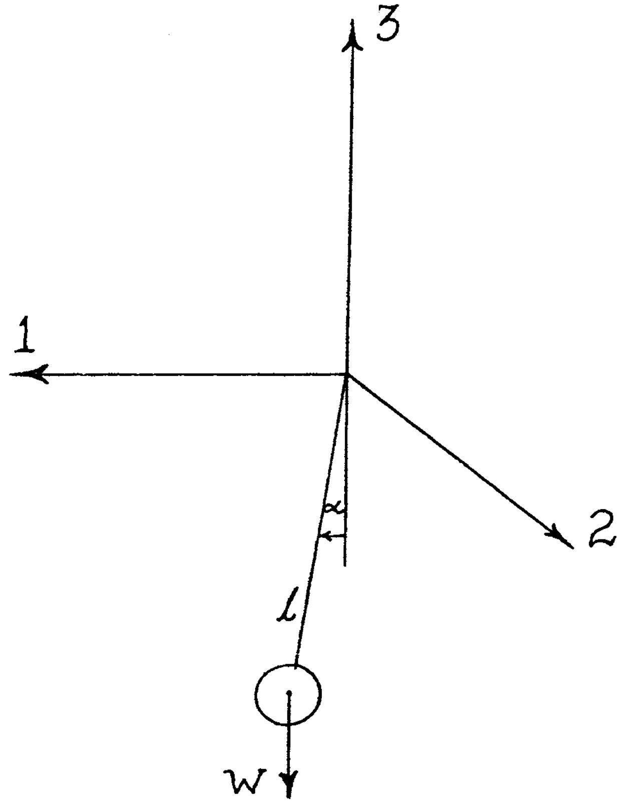


Figure 7.

POTENTIAL ENERGY OF THE PENDULOUS WEIGHT

$R_3 = -D_{56}\dot{\phi}$ = torque damping the angular motion in the ϕ direction

D_{12}, D_{34}, D_{56} = viscous damping coefficients = constants

Considering plane 1-2 as the reference, the potential energy U can be written as,

$$U = -\omega \ell \cos\alpha.$$

Hence, the generalized forces Q_i can be written as,

$$Q_1 = R_1 - \frac{\delta U}{\delta \psi} = R_1 = -D_{12}\dot{\psi}$$
$$Q_2 = R_2 - \frac{\delta U}{\delta \alpha} = -D_{34}\dot{\alpha} - \omega \ell \sin\alpha$$
$$Q_3 = R_3 - \frac{\delta U}{\delta \phi} = -D_{56}\dot{\phi}$$

From (5), the equations of motion are obtained as,

$$\begin{aligned} &\frac{d}{dt}\frac{\delta T}{\delta \dot{\psi}} - \frac{\delta T}{\delta \psi} = -D_{12}\dot{\psi} && \text{(i)} \\ &\frac{d}{dt}\frac{\delta T}{\delta \dot{\alpha}} - \frac{\delta T}{\delta \alpha} = -(D_{34}\dot{\alpha} + \omega \ell \sin\alpha) && \text{(ii)} \\ &\frac{d}{dt}\frac{\delta T}{\delta \dot{\phi}} - \frac{\delta T}{\delta \phi} = -D_{34}\dot{\phi} && \text{(iii)} \end{aligned} \qquad (6)$$

Substituting appropriate terms in the above equations leads to the exact equations of motion.

The various derivatives of the kinetic energy function T, appeari in the equations of motion are obtained as follows:

$$\frac{\delta T}{\delta \psi} = 1/2[-C_o\Omega_c{}^2\sin2\psi + A_o\Omega_c{}^2\sin2\psi - 2(A+A_I)\omega_8\Omega_c\cos\psi$$
$$+ 2(A+B_I)\omega_9\Omega_c\sin\psi\sin\alpha - 2C\omega_7\Omega_c\sin\psi\cos\alpha$$

$$- 2C_I\omega_7^*\Omega_c\sin\psi\cos\alpha + 2m(\ell\omega_7^* - R\Omega_c\cos\psi)\cdot$$
$$(-\ell\Omega_c\cos\alpha\sin\psi + R\Omega_c\sin\psi) + 2m(R\Omega_c\sin\alpha\sin\psi - e\omega_7^*)\cdot$$
$$(R\Omega_c\sin\alpha\cos\psi + e\Omega_c\sin\psi\cos\alpha) + 2m(e\omega_9 - \ell\omega_8 - R\Omega_c\sin\psi\cos\alpha)\cdot$$
$$(e\Omega_c\sin\psi\sin\alpha + \ell\Omega_c\cos\psi - R\Omega_c\cos\psi\cos\alpha)]$$

Rearranging,

$$\frac{\delta T}{\delta\psi} = 1/2[\Omega_c^{\ 2}\sin2\psi(A_o - C_o) - 2(A+A_I)\omega_8\cdot\Omega_c\cdot\cos\psi + 2\Omega_c\sin\psi$$
$$\{(A+B_I)\omega_9\cdot\sin\alpha - \cos\alpha(C\omega_7 + C_I\omega_7^*)\} + 2m\{\Omega_c\sin\psi$$
$$(\ell\omega_7^* - R\Omega_c\cos\psi)\cdot(R - \ell\cos\alpha) + (R\Omega_c\sin\alpha\sin\psi - e\omega_7^*)\cdot$$
$$(R\Omega_c\sin\alpha\cos\psi + e\Omega_c\sin\psi\cos\alpha) + (e\omega_9 - \ell\omega_8 - R\Omega_c\sin\psi\cos\alpha)\cdot$$
$$(e\Omega_c\sin\psi\sin\alpha + \Omega_c\cos\psi(\ell - R\cos\alpha))\}]$$

$$\frac{\delta T}{\delta\dot{\psi}} = 1/2[2A_o(\dot{\psi}+\Omega_s) + 2(A+B_I)\omega_9\cos\alpha + 2C\omega_7\sin\alpha + 2C_I\omega_7^*\sin\alpha$$
$$+ 2m\{\ell\sin\alpha(\ell\omega_7^* - R\Omega_c\cos\psi) + (R\Omega_c\sin\alpha\sin\psi - e\omega_7^*)\cdot$$
$$(-e\sin\alpha) + (e\omega_9 - \ell\omega_8 - R\Omega_c\sin\psi\cos\alpha)\cdot(e\cos\alpha)\}]$$

Rearranging,

$$\frac{\delta T}{\delta\dot{\psi}} = [A_o(\dot{\psi}+\Omega_s) + (A+B_I)\omega_9\cos\alpha + (C\omega_7 + C_I\omega_7^*)\sin\alpha + m\ell\sin\alpha$$
$$(\ell\omega_7^* - R\Omega_c\cos\psi) - me\sin\alpha(R\Omega_c\sin\alpha\sin\psi - e\omega_7^*) +$$
$$me\cos\alpha(e\omega_9 - \ell\omega_8 - R\Omega_c\sin\psi\cos\alpha)]$$

$$\frac{\delta T}{\delta\alpha} = 1/2[-2(A+B_I)\omega_9\{(\dot{\psi}+\Omega_s)\sin\alpha + \Omega_c\cos\psi\cos\alpha\} + 2(C\omega_7 + C_I\omega_7^*)$$
$$\{(\dot{\psi}+\Omega_s)\cos\psi - \Omega_c\cos\psi\sin\alpha\} + 2m\{(\ell\omega_7^* - R\Omega_c\cos\psi)$$
$$[\ell\cos\alpha(\dot{\psi}+\Omega_s) - \ell\Omega_c\sin\alpha\cos\psi] + (R\Omega_c\sin\alpha\sin\psi - e\omega_7^*)\cdot$$
$$(R\Omega_c\cos\alpha\sin\psi + e\Omega_c\cos\psi\sin\alpha - e(\dot{\psi}+\Omega_s)\cos\alpha) +$$
$$(e\omega_9 - \ell\omega_8 - R\Omega_c\sin\psi\cos\alpha)\cdot[-e(\dot{\psi}+\Omega_s)\sin\alpha$$
$$-e\Omega_c\cos\psi\cos\alpha + R\Omega_c\sin\psi\sin\alpha]\}]$$

Rearranging,

$$\frac{\delta T}{\delta \alpha} = [(C\omega_7 + C_I\omega_7^*)\cdot\{(\dot\psi+\Omega_s)\cos\alpha - \Omega_c\cos\psi\sin\alpha\} - (A+B_I)\omega_9$$
$$\{(\dot\psi+\Omega_s)\sin\alpha + \Omega_c\cos\psi\cos\alpha\} + m(\ell\omega_7^* - R\Omega_c\cos\psi)\cdot$$
$$\{\ell\cos\alpha(\dot\psi+\Omega_s) - \ell\Omega_c\sin\alpha\cos\psi\} + m(R\Omega_c\sin\alpha\sin\psi - e\omega_7^*)\cdot$$
$$\{(R\Omega_c\sin\psi - e(\dot\psi+\Omega_s))\cdot\cos\alpha + e\Omega_c\cos\psi\sin\alpha\} +$$
$$m(e\omega_9 - \ell\omega_8 - R\Omega_c\sin\psi\cos\alpha)\cdot\{(R\Omega_c\sin\psi - e(\dot\psi+\Omega_s))$$
$$\sin\alpha - e\Omega_c\cos\psi\cos\alpha\}]$$

$$\frac{\delta T}{\delta \dot\alpha} = 1/2[2(A+A_I)\omega_8(-1) + 2m(e\omega_9 - \ell\omega_8 - R\Omega_c\sin\psi\cos\alpha)\cdot(-\ell)(-1)]$$

Rearranging,

$$\frac{\delta T}{\delta \dot\alpha} = [m\ell(e\omega_9 - \ell\omega_8 - R\Omega_c\sin\psi\cos\alpha) - (A+A_I)\omega_8]$$

$$\frac{\delta T}{\delta \phi} = 0; \qquad \frac{\delta T}{\delta \dot\phi} = 1/2[2\cdot C\cdot\omega_7(1)] = (C\cdot\omega_7)$$

Before proceeding with the equations of motion, define,

$$H_1 = (\dot\psi+\Omega_s)$$
$$H_2 = (C\omega_7 + C_i\omega_7^*)$$
$$H_3 = (\ell\omega_7^* - R\Omega_c\cos\psi)$$
$$H_4 = (R\Omega_c\sin\alpha\sin\psi - e\omega_7^*)$$
$$H_5 = (e\omega_9 - \ell\omega_8 - R\Omega_c\sin\psi\cos\alpha)$$
$$H_6 = \cos\psi\sin\alpha\cdot\dot\alpha$$
$$H_7 = \sin\psi\cos\alpha\cdot\dot\psi$$
$$H_8 = \Omega_c\cos\psi\cos\alpha$$
$$H_9 = \Omega_c\cos\psi\sin\alpha$$
$$H_{10} = \Omega_c\sin\psi\sin\alpha$$
$$H_{11} = \Omega_c\sin\psi\cos\alpha$$

$$H_{12} = -H_9\cdot\dot{\alpha} - H_{11}\cdot\dot{\psi} + H_1\cdot\cos\alpha\cdot\dot{\alpha}$$

$$H_{13} = \Omega_c^{\,2}(A_o-C_o)\frac{\sin 2\psi}{2} - (A+A_I)\omega_8\cdot\Omega_c\cos\psi - H_2\cdot H_{11} + (A+B_I)\cdot\omega_9\cdot H_{10}$$
$$+ m\{R\Omega_c\sin\psi\cdot H_3 - \ell\cdot H_3\cdot H_{11} + (R\cdot H_{10} - e\omega_7^*)\cdot(R\cdot H_9 + e\cdot H_{11})$$
$$+ H_5(e\cdot H_{10} + \ell\Omega_c\cos\psi - R\cdot H_8)\}$$

$$H_{14} = A_o + (A+B_I)\cos^2\psi + (C+C_I+m\ell^2)\sin^2\alpha + me^2$$

which occur frequently in the equations.

From equation 6(i),

$$\frac{d}{dt}\left(\frac{\delta T}{\delta\dot{\psi}}\right) - \frac{\delta T}{\delta\psi} = -D_{12}\dot{\psi}$$

Substituting, we have

$$\ddot{\psi} = \frac{1}{[A_o + (A+B_I)\cos^2\alpha + (C+C_I+mL^2)\sin^2\alpha + me^2]}\ \mathrm{x}$$

$$[-D_{12}\dot{\psi} + [\Omega_c^{\,2}(A_o-C_o)\frac{\sin 2\psi}{2} - (A+A_I)\omega_8\cdot\Omega_c\cos\psi +$$
$$(A+B_I)\cdot\omega_9\cdot\Omega_c\sin\psi\sin\alpha - (C\omega_7 + C_I\omega_7^*)\cdot(\Omega_c\sin\psi\cos\alpha)$$
$$+ m\{R\Omega_c\sin\psi(\ell\omega_7^* - R\Omega_c\cos\psi) - \ell(\ell\omega_7^* - R\Omega_c\cos\psi)\cdot(\Omega_c\sin\psi\cos\alpha)$$
$$+ (R\Omega_c\sin\psi\sin\alpha - e\omega_7^*)\cdot(R\Omega_c\cos\psi\sin\alpha + e\Omega_c\sin\psi\cos\alpha) +$$
$$(e\omega_9 - \ell\omega_8 - R\Omega_c\sin\psi\cos\alpha)\cdot(e\Omega_c\sin\psi\sin\alpha + \ell\Omega_c\cos\psi - R\Omega_c\cos\psi\cos\alpha)\}]$$
$$- \quad [(A+B_I)\cdot\{-\omega_9\sin\alpha\cdot\dot{\alpha} - (\dot{\psi}+\Omega_s)\sin\alpha\cdot\cos\alpha\cdot\dot{\alpha} - \Omega_c\cos\psi\cos^2\alpha\cdot\dot{\alpha}$$
$$+ \Omega_c\sin\psi\sin\alpha\cos\alpha\cdot\dot{\psi}\} + (C\omega_7 + C_I\omega_7^*)\cdot\cos\alpha\cdot\dot{\alpha} + \sin\alpha$$
$$\{(-\Omega_c\cos\psi\sin\alpha\cdot\dot{\alpha} - \Omega_c\sin\psi\cos\alpha\cdot\dot{\psi} + (\dot{\psi}+\Omega_s)\cdot\cos\alpha\cdot\dot{\alpha})\ \cdot\ (C+C_I) + C\ddot{\phi}\}$$
$$+ m\ell\{(\ell\omega_7^* - R\Omega_c\cos\psi)\cdot\cos\alpha\cdot\dot{\alpha} + \sin\alpha(-\Omega_c\cos\psi\sin\alpha\cdot\dot{\alpha} - \Omega_c\sin\psi\cos\alpha\cdot\dot{\psi}$$
$$+ (\dot{\psi}+\Omega_s\cos\alpha\cdot\dot{\alpha}) + R\Omega_c\sin\alpha\sin\psi\cdot\dot{\psi}\} - me\sin\alpha\{R(\Omega_c\cos\psi\sin\alpha\cdot\dot{\psi}$$
$$+ \Omega_c\sin\psi\cos\alpha\cdot\dot{\alpha}) + e(\Omega_c\cos\psi\sin\alpha\cdot\dot{\alpha} + \Omega_c\sin\psi\cos\alpha\cdot\dot{\psi} - (\dot{\psi}+\Omega_s)\cos\alpha\cdot\dot{\alpha})$$

$$+ (e\omega_9 - \ell\omega_8 - R\Omega_c \sin\psi\cos\alpha)\dot{\alpha}\} + me\cos\alpha\{e(-(\dot{\psi}+\Omega_s)\sin\alpha\cdot\dot{\alpha}$$

$$- \Omega_c\cos\psi\cos\alpha\cdot\dot{\alpha} + \Omega_c\sin\psi\sin\alpha\cdot\dot{\psi}) + \ell(\ddot{\alpha}+\Omega_c\cos\psi\cdot\dot{\psi}) +$$

$$R(\Omega_c\sin\psi\sin\alpha\cdot\dot{\alpha} - \Omega_c\cos\psi\cos\alpha\cdot\dot{\psi}) - (R\Omega_c\sin\alpha\sin\psi - e\omega_7^*)\dot{\alpha}\}]]$$

or substituting H_1, H_2, H_3, etc., we have,

$$\ddot{\psi} = \frac{1}{H_{14}} [-D_{12}\cdot\dot{\psi} + H_{13} - [(A+B_I)\cdot\{-\omega_9\sin\alpha\cdot\dot{\alpha} - H_1\cdot\sin\alpha\cdot\dot{\alpha} - H_8\cdot\dot{\alpha}$$

$$+ H_{10}\cdot\dot{\psi}\}\cos\alpha + H_2\cdot\cos\alpha + \sin\alpha \cdot \{(C+C_I)\cdot H_{12} + C\ddot{\phi}\} + m\ell$$

$$\cdot (H_3\cdot\cos\alpha\cdot\dot{\alpha} + \sin\alpha\cdot H_{12} + R\cdot H_{10}\cdot\dot{\psi}) - m\cdot e\cdot\sin\alpha\cdot$$

$$\{R\cdot(H_9\cdot\dot{\psi} + H_{11}\cdot\dot{\alpha}) + e\cdot(H_9\cdot\dot{\alpha} + H_{11}\cdot\dot{\psi} - H_1\cdot\cos\alpha\cdot\dot{\alpha}) + (H_5\cdot\dot{\alpha})\}$$

$$+ m\cdot e\cdot\cos\alpha\cdot\{e\cdot(-H_1\cdot\sin\alpha\cdot\dot{\alpha} - H_8\cdot\dot{\alpha} + H_{10}\cdot\dot{\psi}) + \ell\cdot(\ddot{\alpha}+\Omega_c\cdot\cos\psi\cdot\dot{\psi})$$

$$+ R\cdot(H_{10}\cdot\dot{\alpha} - H_8\cdot\dot{\psi}) - H_4\cdot\dot{\alpha}\}]] \tag{I}$$

From equation 6(ii),

$$\frac{d}{dt}\left(\frac{\delta T}{\delta\dot{\alpha}}\right) - \frac{\delta T}{\delta\alpha} = -(w\ell\sin\alpha + D_{34}\cdot\dot{\alpha})$$

Substituting, we have,

$$\ddot{\alpha} = \cancel{-\Omega_c\dot{\psi}\cos\psi} + \frac{1}{(m\ell^2 + A + A_I)} [-\cancel{(A+B_I)}\omega_9\cancel{\{(\dot{\psi}+\Omega_s)\cdot\sin\alpha} +$$

$$\cancel{\Omega_c\cos\psi\cos\alpha}\} + \{(\dot{\psi}+\Omega_s)\cos\alpha - \Omega_c\cos\psi\sin\alpha\}\cdot\{(\cancel{C\omega_7 + C_I\cdot\omega_7^*})$$

$$+ m\ell(\ell\omega_7^* - R\Omega_c\cos\psi)\} + m(R\Omega_c\sin\alpha\sin\psi - e\cdot\omega_7^*)\cdot\{[R\Omega_c\sin\psi \cdot$$

$$- e(\cancel{\dot{\psi}+\Omega_s})]\cancel{\cdot\cos}\alpha + e\cdot\cancel{\Omega_c\cos}\psi\cancel{\sin\alpha}\} + m(e\cdot\omega_9 - \ell\cdot\omega_8 - R\Omega_c\sin\psi\cos\alpha)$$

$$\cdot \{[R\Omega_c\sin\psi - e(\dot{\psi}+\Omega_s)]\sin\alpha - e\Omega_c\cos\psi\cos\alpha\} - m\ell\{e[\ddot{\psi}\cos\alpha - (\dot{\psi}+\Omega_s)$$

$$\cdot\ \sin\alpha\cdot\dot{\alpha} - \Omega_c\cos\psi\cos\alpha\cdot\dot{\alpha} + \Omega_c\sin\psi\sin\alpha\cdot\dot{\psi}] + R\Omega_c\cos\psi$$

$$(\sin\alpha\cdot\dot{\alpha} - \cos\psi\cdot\dot{\psi})\} - \cancel{w\ell\sin\alpha - D_{34}\dot{\alpha}}$$

or, substituting H_1, H_2, H_3, etc., we have,

$$\begin{aligned}\ddot{\alpha} = {} & -\Omega_c\cdot\dot{\psi}\cdot\cos\psi + \frac{1}{(m\ell^2 + A + A_I)}\cdot\ [-(A+B_I)\cdot\omega_9\cdot(H_1\cdot\sin\alpha + H_8) \\ & + (H_1\cdot\cos\alpha - H_9)\cdot(H_2 + m\cdot\ell\cdot H_3) + m\cdot H_4\cdot(R\cdot H_{11} - e\cdot H_1\cdot\cos\alpha + e\cdot H_9) \\ & + m\cdot H_5\cdot(R\cdot H_{10} - e\cdot H_1\cdot\sin\alpha - e\cdot H_8) - m\cdot\ell\cdot\{e\cdot(\ddot{\psi}\cdot\cos\alpha - H_1\cdot\sin\alpha\cdot\dot{\alpha} \\ & - H_8\cdot\Omega_c\cdot\dot{\alpha} + H_{10}\cdot\Omega_c\cdot\dot{\psi}) + R\cdot H_9\cdot\dot{\alpha} - R\cdot H_8\cdot\dot{\psi}\} - w\cdot\ell\cdot\sin\alpha - D_{34}\cdot\dot{\alpha}]\end{aligned}$$

(II)

From equation 6(iii),

$$\frac{d}{dt}\left(\frac{\delta T}{\delta\dot{\phi}}\right) - \frac{\delta T}{\delta\phi} = -D_{56}\dot{\phi}$$

Substituting,

$$C[-\Omega_c\cos\psi\sin\alpha\cdot\dot{\alpha} - \Omega_c\sin\psi\cos\alpha\cdot\dot{\psi} + (\dot{\psi}+\Omega_s)\cos\alpha\cdot\dot{\alpha} + \ddot{\psi}\sin\alpha + \ddot{\phi}] = -D_{56}\dot{\phi}$$

or substituting H_1, H_2, H_3, etc., we have,

$$\ddot{\phi} = \frac{-D_{56}\cdot\dot{\phi}}{C} - H_{12} - \ddot{\psi}\cdot\sin\alpha \qquad \text{(III)}$$

Equations I, II, and III are the exact equations of motion for the gyrocompass.

H. APPROXIMATE DIFFERENTIAL EQUATIONS OF MOTION

If certain assumptions are made to simplify the equations of motion it is possible to obtain approximate solutions to the differential equations. The exact equations are coupled, second order, and non-linear. Hence it is impossible to solve the equations analytically. However, a numerical solution to the differential equations can be obtained. This shall be discussed in a later section.

The exact equations of motion are simplified by assuming:

1. No damping exists at the bearings.

$$D_{12} = D_{34} = D_{56} = 0$$

2. The precession and nutation angles, ψ and α, respectively, are small angles. Hence,

$$\sin \psi \simeq \psi$$

$$\cos \psi \simeq 1$$

$$\sin \alpha \simeq \alpha$$

$$\cos \alpha \simeq 1,$$

3. The spin rate is constant.

$$\dot{\phi} = \text{constant}$$

$$\ddot{\phi} = 0$$

4. The moments of inertia of the outer gimbal, the inner gimbal, and the pendulous weight are ignored.

Applying these approximations to the three equations of motion and dropping the non-linear terms from the resulting equations gives the simplified, linear form of the set of two equations shown below.

$$A\ddot{\psi} + C\dot{\phi}\dot{\alpha} + C\Omega_c\dot{\phi}\psi = 0$$

$$-A\ddot{\alpha} + C\dot{\phi}\dot{\psi} - (w\ell + C\dot{\phi}\Omega_c)\alpha = -C\dot{\phi}\Omega_s$$

These equations can be solved analytically as shown in the next section.

I. EXACT SOLUTION TO APPROXIMATE EQUATIONS OF MOTION

The approximate equations of motion are

$$\ddot{\psi} + \frac{C\dot{\phi}}{A}\,\dot{\alpha} + \frac{a}{A}\,\psi = 0 \quad ; \quad a = C\dot{\phi}\Omega_c$$

$$\ddot{\alpha} - \frac{C\dot{\phi}}{A}\,\dot{\psi} + \frac{b}{A}\,\alpha = \frac{C\dot{\phi}\Omega_s}{A} \; ; \; b = w\ell + C\dot{\phi}\Omega_c$$

the form of which is

$$\ddot{\psi} + \lambda_1\dot{\alpha} + \lambda_2\psi = 0$$

$$\ddot{\alpha} - \lambda_3\dot{\psi} + \lambda_4\alpha = \lambda_5$$

where λ_1, λ_2, λ_3, λ_4, λ_5, are constants.

The second equation is non-homogeneous, the particular solution of which is given by

$$\alpha_{part} = \lambda_5/\lambda_4$$

The complementary solution is found as follows:

$$\begin{aligned} \ddot{\psi} + \lambda_1\dot{\alpha} + \lambda_2\psi &= 0 \\ \ddot{\alpha} - \lambda_3\dot{\psi} + \lambda_4\alpha &= 0 \end{aligned} \tag{7}$$

Assume solutions of the form

$$\begin{aligned} \psi &= R\, e^{st} \\ \alpha &= D\, e^{st} \end{aligned} \tag{8}$$

where R and D are undetermined constants.

Substituting (8) into (7)

$$\begin{aligned} (s^2+\lambda_2)R\cancel{e^{st}} + s\lambda_1 D\cancel{e^{st}} &= 0 \\ -s\lambda_3 R\cancel{e^{st}} + (s^2+\lambda_4)D\cancel{e^{st}} &= 0 \end{aligned} \tag{9}$$

Equating the determinant of the left hand side to zero will yield the frequencies of oscillation s.

$$\begin{vmatrix} (s^2+\lambda_2) & s\lambda_1 \\ -s\lambda_3 & (s^2+\lambda_4) \end{vmatrix} = 0$$

$$(s^2+\lambda_2)(s^2+\lambda_4) + s^2\lambda_1\lambda_3 = 0$$

$$s^4 + (\lambda_2 + \lambda_4 + \lambda_1\lambda_3)s^2 + \lambda_2\lambda_4 = 0$$

$$s^2 = \frac{-(\lambda_2+\lambda_4+\lambda_1\lambda_3) \pm\sqrt{(\lambda_2+\lambda_4+\lambda_1\lambda_3)^2-4\lambda_2\lambda_4}}{2}$$

As $(\lambda_2+\lambda_4+\lambda_1\lambda_3)$ and $4\lambda_2\lambda_4$ are both positive, and $(\lambda_2+\lambda_4+\lambda_1\lambda_3) >> 4\lambda_2\lambda_4$,[(2)] s^2 will always be negative.

[(2)] The values of the parameters used are as follows:

$$\phi = 1000 \text{ radians/second}$$

$$C = 3.0 \text{ in. lb. sec.}^2$$

$$A = 1.8 \text{ in. lb. sec.}^2$$

$$\Omega = 7.27 \times 10^{-5} \text{ radians/second}$$

$$\lambda = 38°$$

$$w = 15.0 \text{ lbs.}$$

$$\ell = 5.0 \text{ in.}$$

$$\lambda_1 = \frac{C\dot{\phi}}{A} = 1667.0$$

$$\lambda_2 = \frac{C\dot{\phi}\Omega_c}{A} = 0.0955$$

$$\lambda_3 = \frac{C\dot{\phi}}{A} = 1667.0$$

$$\lambda_4 = \frac{(w\ell+C\dot{\phi}\Omega_c)}{A} = 41.7955$$

(2) (continued)

$$\lambda_5 = \frac{C\dot{\phi}\Omega_s}{A} = 0.0747$$

Thus $(\lambda_2+\lambda_4+\lambda_1\lambda_3)$, $4\lambda_2\lambda_4$ are positive; and $(\lambda_2+\lambda_4+\lambda_1\lambda_3) >> 4\lambda_2\lambda_4$

Hence the roots s will always be imaginary.

If

$$p_1 = \sqrt{\frac{(\lambda_2+\lambda_4+\lambda_1\lambda_3) - \sqrt{(\lambda_2+\lambda_4+\lambda_1\lambda_3)^2 - 4\lambda_2\lambda_4}}{2}}$$

and

$$p_2 = \sqrt{\frac{(\lambda_2+\lambda_4+\lambda_1\lambda_3) + \sqrt{(\lambda_2+\lambda_4+\lambda_1\lambda_3)^2 - 4\lambda_2\lambda_4}}{2}}$$

then the roots s are given by

$$\begin{aligned} s_1 &= ip_1 \\ s_2 &= -ip_1 \\ s_3 &= ip_2 \\ s_4 &= -ip_2 \end{aligned} \qquad (10)$$

Substitute (10) into (9) in order to obtain the amplitude ratios R_i/D_i.

From (9),

$$\frac{D}{R} = \frac{-(s^2+\lambda_2)}{s\lambda_1} = \frac{s\lambda_3}{(s^2+\lambda_4)}$$

Hence,

$$\frac{D_1}{R_1} = \frac{ip_1\lambda_3}{-p_1^2+\lambda_4} \qquad \text{(i)}$$

$$\frac{D_2}{R_2} = \frac{-ip_1\lambda_3}{-p_1^2+\lambda_4} \qquad \text{(ii)}$$

$$\frac{D_3}{R_3} = \frac{ip_2\lambda_3}{-p_2^2+\lambda_4} \qquad \text{(iii)} \qquad (11)$$

$$\frac{D_4}{R_4} = \frac{-ip_2\lambda_3}{-p_2^2+\lambda_4} \qquad \text{(iv)}$$

The general solution (8) will take the form

$$\psi_{(t)} = R_1e^{ip_1t} + R_2e^{-ip_1t} + R_3e^{ip_2t} + R_4e^{-ip_2t} \tag{12}$$

$$\alpha_{(t)} = D_1e^{ip_1t} + D_2e^{-ip_1t} + D_3e^{ip_2t} + D_4e^{-ip_2t} \tag{13}$$

In equation (12), let

$$R_1e^{ip_1t} + R_2e^{-ip_1t} = M_1\sin p_1t + M_2\cos p_1t$$

where M_1, M_2 are real constants. (14)

$$R_1(\cos p_1t + i \sin p_1t) + R_2(\cos p_1t - i \sin p_1t)$$
$$= M_1\sin p_1t + M_2\cos p_1t$$

Equating the real and imaginary part on both sides,

$$\begin{aligned} R_1 + R_2 &= M_2 \\ (R_1 - R_2)i &= M_1 \end{aligned} \tag{15}$$

Similarly introduce other constants M_3, M_4, N_1, N_2, N_3 and N_4, whence from equation (13),

$$D_1e^{ip_1t} + D_2e^{-ip_1t} = N_1\sin p_1t + N_2\cos p_1t$$

$$\begin{aligned} D_1 + D_2 &= N_2 \\ (D_1 - D_2)i &= N_1 \end{aligned} \tag{16}$$

Similarly,

$$\begin{aligned} R_3 + R_4 &= M_4 \\ (R_3 - R_4)i &= M_3 \end{aligned} \tag{17}$$

$$\begin{aligned} D_3 + D_4 &= N_4 \\ (D_3 - D_4)i &= N_3 \end{aligned} \tag{18}$$

Now, $$\frac{M_1}{N_2} = \frac{(R_1 - R_2)i}{(D_1 + D_2)}$$

From 11(i) and 11(ii), it can be seen that

$$\frac{R_1}{D_1} = -\frac{R_2}{D_2}$$

$$\frac{R_3}{D_3} = -\frac{R_4}{D_4} \tag{19}$$

Hence

$$\frac{M_1}{N_2} = \frac{[-\frac{R_2 D_1}{D_2} - R_2]i}{(D_1 + D_2)} = \frac{-iR_2}{D_2}$$

$$\frac{M_1}{N_2} = \frac{-iR_2}{D_2} = \frac{(-p_1^2 + \lambda_4)}{p_1 \lambda_3}$$

Now $$\frac{M_1}{N_2} = \frac{R_1 + R_2}{(D_1 - D_2)i} \tag{20}$$

Substituting from (19) for R_2,

$$\frac{M_2}{N_1} = \frac{(R_1 - \frac{R_1 D_2}{D_1})}{(D_1 - D_2)i} = \frac{R_1}{D_1 i} = \frac{R_1}{D_1 i}$$

$$\frac{M_2}{N_1} = \frac{(-p_1^2 + \lambda_4)}{i^2 p_1 \lambda_3} = \frac{(p_1^2 - \lambda_4)}{p_1 \lambda_3} \tag{21}$$

$$\frac{M_2}{N_1} = \frac{-M_1}{N_2} \tag{22}$$

Similarly, $$\frac{M_3}{N_4} = \frac{(-p_2^2 + \lambda_4)}{p_2 \lambda_3} \tag{23}$$

$$\frac{M_4}{N_3} = \frac{(p_2^2 - \lambda_4)}{p_2 \lambda_3} \tag{24}$$

$$\frac{M_3}{N_4} = \frac{-M_4}{N_3} \tag{25}$$

From (12), (13) and (14), the solutions take the form

$$\psi_{(t)} = M_1\sin\, p_1t + M_2\cos\, p_1t + M_3\sin\, p_2t + M_4\cos\, p_2t \tag{26}$$

$$\alpha_{(t)} = N_1\sin\, p_1t + N_2\cos\, p_1t + N_3\sin\, p_2t + N_4\cos\, p_2t \tag{27}$$

where M_i, N_i, p_i are all real.

Substituting for the M_i's in terms of the N_i's from equations (20) through (25) into equations (26) and (27),

$$\begin{aligned}\psi_{(t)} = {} & \frac{(-p_1{}^2+\lambda_4)}{p_1\lambda_3} N_2\sin\, p_1t + \frac{(p_1{}^2-\lambda_4)}{p_1\lambda_3} N_1\cos\, p_1t + \\ & \frac{(-p_2{}^2+\lambda_4)}{p_2\lambda_3} N_4\sin\, p_2t + \frac{(p_2{}^2-\lambda_4)}{p_2\lambda_3} N_3\cos\, p_2t\end{aligned} \tag{28}$$

$$\alpha_{(t)} = N_1\sin\, p_1t + N_2\cos\, p_1t + N_3\sin\, p_2t + N_4\cos\, p_2t + \frac{\lambda_5}{\lambda_4} \tag{29}$$

The constants N_i can be determined from initial conditions. For example, a set of initial conditions are chosen as follows:

$$\begin{aligned}\psi_{(0)} &= 0.1 \quad ; \quad \alpha_{(0)} = 0.1 \\ \dot{\psi}_{(0)} &= 0 \quad\;\; ; \quad \dot{\alpha}_{(0)} = 0\end{aligned} \tag{30}$$

which are the same conditions as the ones used in the numerical solution of the exact equations.

Substituting (30) into (28) and (29),

$$\psi_{(0)} = \frac{(p_1{}^2-\lambda_4)}{p_1\lambda_3} N_1 + \frac{(p_2{}^2-\lambda_4)}{p_2\lambda_3} N_3 = 0.1$$

$$N_1 = [(0.1)\lambda_3 - \frac{(p_2^{\ 2}-\lambda_4)}{p_2} N_3] \frac{p_1}{(p_1^{\ 2}-\lambda_4)}$$

$$\alpha_{(0)} = N_2 + N_4$$

$$= 0.1 - \alpha_{particular} = 0.1 - \frac{\lambda_5}{\lambda_4}$$

$$N_2 = 0.1 - N_4 - \frac{\lambda_5}{\lambda_4}$$

$$\dot{\psi}_{(0)} = \frac{(-p_1^{\ 2}+\lambda_4)N_2}{\lambda_3} + \frac{(-p_2^{\ 2}+\lambda_4)N_4}{\lambda_3} = 0$$

$$(-p_1^{\ 2}+\lambda_4)(0.1 - N_4 - \frac{\lambda_5}{\lambda_4}) + (-p_2^{\ 2}+\lambda_4)N_4 = 0$$

$$N_4[(p_1^{\ 2}-\lambda_4) + (-p_2^{\ 2}+\lambda_4)] + (0.1 - \frac{\lambda_5}{\lambda_4})(-p_1^{\ 2}+\lambda_4) = 0$$

$$N_4(p_1^{\ 2}-p_2^{\ 2}) = (p_1^{\ 2}-\lambda_4)(0.1 - \frac{\lambda_5}{\lambda_4})$$

$$N_4 = \frac{(p_1^{\ 2}-\lambda_4)(0.1 - \frac{\lambda_5}{\lambda_4})}{(p_1^{\ 2}-p_2^{\ 2})} \tag{31}$$

$$N_2 = 0.1 - N_4 - \frac{\lambda_5}{\lambda_4} \tag{32}$$

$$\dot{\alpha}_{(0)} = N_1p_1 + N_3p_2 = 0$$

$$[(0.1)\lambda_3p_2 - (p_2^{\ 2}-\lambda_4)N_3]\frac{p_1^{\ 2}}{p_2(p_1^{\ 2}-\lambda_4)} + N_3p_2 = 0$$

$$N_3[p_2 - \frac{p_1^{\ 2}(p_2^{\ 2}-\lambda_4)}{p_2(p_1^{\ 2}-\lambda_4)}] = \frac{-(0.1)\lambda_3p_2p_1^{\ 2}}{p_2(p_1^{\ 2}-\lambda_4)}$$

$$N_3 \frac{[p_2^{\,2}(p_1^{\,2}-\lambda_4) - p_1^{\,2}(p_2^{\,2}-\lambda_4)]}{p_2(p_1^{\,2}-\lambda_4)} = \frac{-0.1\ \lambda_3 p_1^{\,2}}{(p_1^{\,2}-\lambda_4)}$$

$$N_3 \frac{(p_1^{\,2}-p_2^{\,2})\lambda_4}{p_2(p_1^{\,2}-\lambda_4)} = \frac{-(0.1)\lambda_3 \cdot p_1^{\,2}}{(p_1^{\,2}-\lambda_4)}$$

$$N_3 = \frac{-(0.1)p_2(\lambda_3)p_1^{\,2}}{(p_1^{\,2}-p_2^{\,2})\lambda_4} \tag{33}$$

$$N_1 = [(0.1)\lambda_3 - \frac{(p_2^{\,2}-\lambda_4)}{p_2} N_3] \frac{p_1}{(p_1^{\,2}-\lambda_4)}$$

or

$$N_1 = -N_3 \frac{p_2}{p_1} \tag{34}$$

Equations (31), (32), (33), and (34) give the values of the constants N_1, N_2, N_3, N_4, which when substituted in (28) and (29) give the solution of the equations.

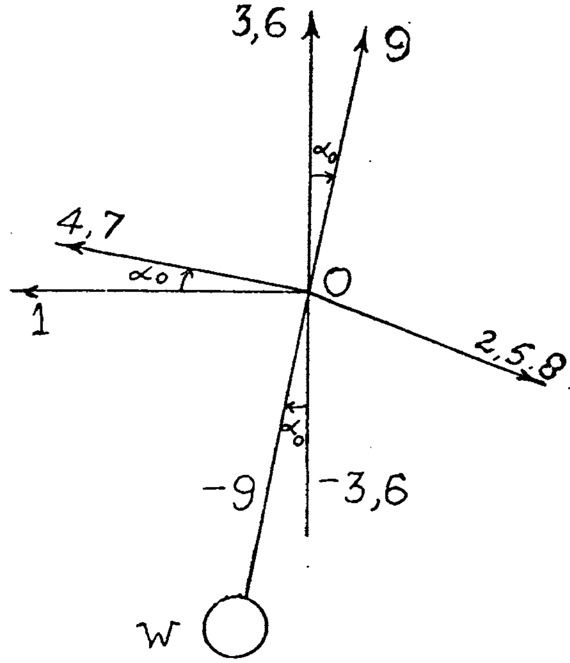


Figure 8.

THE EQUILIBRIUM CONFIGURATION

III. EQUILIBRIUM CONFIGURATION

From the approximate solution of the differential equations of motion, it is seen that the precessional and the nutational motion depends upon the latitude and the time. But this is for a general motion at any time t. It would be interesting to find the position of the gyro axis at a particular time, when the instrument is in a state of equilibrium and the only motion of the system is rotation of the rotor about the 7 axis at a constant rate. In other words, an attempt is made to find the inclination α of the rotor axis with the true North direction in the horizontal plane (i.e., the position where $\alpha = 0$ radian) at an instant when the angular displacement in the ψ direction is zero. This, then shall be the undisturbed condition of the gyrocompass with a perfectly balanced rotor. The system at this instant is said to be in equilibrium.

When in equilibrium, the instrument acquires the position shown in Figure 8. Axis 1 points to the true North in the meridian plane. The rotor axis 7 is displaced from the 1-axis by an angle α_o (the value of α in the equilibrium state), which is constant. Axes 3 and 6 point towards the local vertical. The 9-axis is displaced from the 3-axis by the same angle α_o. Axes 2, 5, and 8 point towards the West. Axes 1, 3, 4, 6, 7, 9 are all co-planar. Since the instrument is in equilibrium, there is no relative motion between the 123 and the 789 coordinate axes.

The relationship between the sets of axes 123 and 789 for the equilibrium configuration can be obtained from section I-D, equations (4), by substituting $\psi = 0$ and $\alpha = \alpha_o$ as follows

$$\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \begin{bmatrix} \cos\alpha_o & 0 & -\sin\alpha_o \\ 0 & 1 & 0 \\ \sin\alpha_o & 0 & \cos\alpha_o \end{bmatrix} \begin{bmatrix} \hat{e}_7 \\ \hat{e}_8 \\ \hat{e}_9 \end{bmatrix} \qquad (34)$$

For the equilibrium state, it is logical to assume that

1. No damping exists. $D_{12} = D_{34} = D_{56} = 0$
2. There is no motion in the direction of ψ. $\dot{\psi} = \ddot{\psi} = 0$
3. The gyro is spinning at a constant rate. $\dot{\phi}$ = constant

   $\ddot{\phi} = 0$
4. α_o is small and $\dot{\alpha} = \ddot{\alpha} = 0$
5. The rotation of the earth from West to East, indicated by

   $$\Omega\hat{k} = \left(\frac{2\pi}{24 \times 3600}\right)\hat{k}$$

   $$\Omega = 7.27 \times 10^{-5} \text{ radians/second}$$

   is small.

With these assumptions the angular velocity of the 789 axes, $\overline{\omega}_{789}$, as obtained in section I-E, becomes

$$\omega_7^* = \Omega_c\cos\alpha_o + \Omega_s\sin\alpha_o = \Omega\cos(\lambda-\alpha_o)$$

$$\omega_8 = 0$$

$$\omega_9 = \Omega_s\cos\alpha_o - \Omega_c\sin\alpha_o = \Omega\cdot\sin(\lambda-\alpha_o)$$

Now, consider the equation for $\ddot{\alpha}$, from section I-G, which is the equation of interest. Rewriting it after substituting the approximations for the equilibrium configuration,

$$\ddot{\alpha} = \frac{1}{(m\ell^2+A+A_I)} \cdot [-(A+B_I)\omega_9(\Omega_s \sin\alpha_o + \Omega_c \cos\alpha_o) +$$

$$(\Omega_s \cos\alpha_o - \Omega_c \sin\alpha_o)\cdot\{C\omega_7 + C_I\omega_7{}^* + m\ell(\ell\omega_7{}^* - R\Omega_c)\} - me\omega_7^*\{-e\Omega_s \cos\alpha_o$$

$$+ e\Omega_c \sin\alpha_o\} + me\omega_9\,\{-e\Omega_s \sin\alpha_o - e\Omega_c \cos\alpha_o\} - w\ell \sin(\alpha_o)] = 0 \quad (35)$$

But,

$$(\Omega_c \cos\alpha_o + \Omega_s \sin\alpha_o) = \omega_7{}^*, \text{ and}$$

$$(\Omega_s \cos\alpha_o - \Omega_c \sin\alpha_o) = \omega_9.$$

Hence, from equation (35),

$$[(C+C_I) - (A+B_I)]\omega_7^*\omega_9 + C\dot{\phi}\omega_9 + m\ell\omega_9(\ell\omega_7^* - R\Omega_c) + me^2\omega_7^*\omega_9 -$$

$$me^2\omega_7^*\omega_9 - w\ell\sin\alpha_o = 0 \quad (36)$$

In the above expression, terms like $\omega_7{}^*\omega_9$ and $m\ell R\omega_9 \cdot \Omega_c$ contain the square of Ω. Ω being of order 10^{-5}, Ω^2 will be of the order 10^{-10}, which is extremely small, and can be neglected, so that equation (36) now becomes,

$$C\dot{\phi}\omega_9 = w\ell\sin\alpha_o$$

Substituting for ω_9,

$$C\dot{\phi}(\Omega_s \cos\alpha_o - \Omega_c \sin\alpha_o) = w\ell\sin\alpha_o$$

Rearranging,

$$\sin\alpha_o(w\ell + C\dot{\phi}\Omega_c) = \cos\alpha_o(C\dot{\phi}\Omega_s)$$

$$\tan\alpha_o(w\ell + C\dot{\phi}\Omega_c) = C\dot{\phi}\Omega_s$$

$$\tan\alpha_o = \frac{C\dot{\phi}\Omega_s}{w\ell + C\dot{\phi}\Omega_c}$$

$$\alpha_o = \tan^{-1} \left[\frac{C\dot{\phi}\Omega\sin\lambda}{w\ell + C\dot{\phi}\Omega\cos\lambda}\right]$$

which gives the expression for the inclination of the rotor axis with the horizon. This angle depends upon the latitude, λ.

IV. NUMERICAL SOLUTION OF THE EXACT EQUATIONS OF MOTION

The three equations of motion obtained in part I, section G, are simultaneous and non-linear, and hence too complicated to be solved analytically. Hence, it is sought to solve these equations on a digital computer, using the numerical methods available for solving differential equations. The most commonly used numerical methods for solving differential equations are the Runge-Kutta Methods, the Multi-step Formulas, and the Predictor-Corrector methods. The methods used in solving the present problem are (i) the Runge-Kutta method of order 4 and (ii) the Hamming's method, which is a Predictor-Corrector type of method. The computer used is an IBM 360 model. A brief description of the two methods follows.

The Runge-Kutta Method of Order 4:

The formula used in this method is

$$y_{n+1} = y_n + 1/6(k_1 + 2k_2 + 2k_3 + 4k) \qquad (37)$$

where

$$k_1 = h\cdot f(x_n, y_n)$$
$$k_2 = h\cdot f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$$
$$k_3 = h\cdot f(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$$
$$k_4 = h\cdot f(x_n + h, y_n + k_3)$$

which provides a solution for an equation of the form

$$y' = f(x,y)$$

'h' is the step size

$y_{(x_o)} = y_o$ is the initial condition

The error of the Runge-Kutta formula of order 4 due to truncation is of the order $(h)^5$. The method is self-starting and it requires four derivative evaluations per step.

To apply this method to our system of differential equations, it is necessary to convert the equations, which are of the second order, to the first order, as the Runge-Kutta formula used here is for first-order equations. For this, an approach of representing m second-order equations as 2m first-order equations can be used. Thus,

$$\ddot{\psi} = \frac{d^2\psi}{dt^2}, \quad \ddot{\alpha} = \frac{d^2\alpha}{dt^2}, \text{ and } \ddot{\phi} = \frac{d^2\phi}{dt^2}$$

can be represented as

$$\ddot{\psi} = \frac{d\dot{\psi}}{dt}, \text{ and } \dot{\psi} = \frac{d\psi}{dt};$$

$$\ddot{\alpha} = \frac{d\dot{\alpha}}{dt}, \text{ and } \dot{\alpha} = \frac{d\alpha}{dt};$$

$$\ddot{\phi} = \frac{d\dot{\phi}}{dt}, \text{ and } \dot{\phi} = \frac{d\phi}{dt}, \text{ respectively.}$$

Thus, we obtain 6 first-order equations from 3 second-order equations. The program written to solve these 6 equations simultaneously appears in the appendix along with the sample results and the graphs.

The Hamming Method of Order 1:

This is a Predictor-Corrector method given by the set of formulas,

$$p_{n+1} = y_{n-3} + \frac{4h}{3} (2y'_n - y'_{n-1} + 2y'_{n-2})$$

$$m_{n+1} = p_{n+1} - \frac{112}{121} (p_n - c_n)$$

$$m'_{n+1} = f(x_{n+1}, m_{n+1})$$

$$C_{n+1} = \frac{1}{8}(9y_n - y_{n-2} + 3h(m'_{n+1} + 2y'_n - y'_{n-1}))$$

$$y_{n+1} = C_{n+1} + \frac{9}{121}(p_{n+1} - C_{n+1})$$

The first equation of the set is the predictor, the second and the third are the modifiers, the fourth is the corrector, and the fifth is the final equation. This method needs 4 starting values. The error at every step is approximated by $[p_{n+1} - C_{n+1}]$. The error term for the 4 starting values, $[p_n - C_n]$ is zero. The 4 starting values can be obtained by using the Runge-Kutta method for the first 4 steps. The computer program for solving the equations of motion using this method is included in the appendix.

V. RESULTS

A. GENERAL

The main objective of this thesis is to make a comparison between (i) the solution to the exact differential equations of motion, obtained numerically and the approximate analytical solution obtained after making several assumptions and linearizing the equations of motion to a simpler form, and (ii) the accuracy and stability of the two numerical methods, namely, the Runge-Kutta method of order 4 and the Hamming method of order 1 used to solve the differential equations of motion.

After determining the suitability of the methods for the type of equations appearing in the present problem the graphs of different variables remain to be plotted.

B. COMPARISON BETWEEN THE RUNGE-KUTTA METHOD AND THE HAMMING METHOD

The Runge-Kutta method of order 4 being self-starting and fairly accurate was the first choice for use in solving the exact equations of motion numerically. The constant parameters like the moments of inertia, angular velocity, weight, length, and eccentricity of the pendulous weight were obtained from the specifications of a gyrocompass in every-day use. Initial conditions for the displacements and the velocities were chosen arbitrarily. The equations being functions of the independent variable time, the time interval Δt between two consecutive readings was the step-size h for the Runge-Kutta method.

The equations were solved several times using a different step-size for every trial. It was observed that the results varied with different time intervals, although the time intervals used were as small as 0.001, 0.0005, and 0.0001 second. The general pattern followed by the results was that the values of ψ increased with time and those of α decreased with increase of time. The values of ψ were increasing uniformly, the values of α showed an oscillatory nature while they decreased. In their absolute values, the results for different time intervals did not match closely with each other.

The Hamming method was selected as another attempt in solving the exact equations numerically. The four starting values were obtained from the results of the Runge-Kutta method using the same step-size. The Hamming method also showed a disagreement amongst the results for different time intervals. Moreover, these results were not comparable to the results of the Runge-Kutta method. The general pattern followed by the Hamming method was that with the increase of time, ψ increased uniformly and α decreased uniformly.

The results thus obtained from both the methods exhibit the phenomenon of "Numerical Instability". Both the Runge-Kutta method and the Hamming method are dependable for simple first order differential equations. But when they are applied to a system of equations such as the present one, which involves one quantity, $\dot{\phi}$ assuming a value as high as 1000 rad./sec. and another, Ω as low as 7.27×10^{-5} rad./sec, their accuracy is diminished unless some special techniques are used to eliminate the errors in the computation.

The computer time used by the Runge-Kutta method was about twice as much as that required by the Hamming method. However, the Runge-Kutta method had the advantage of being self-starting.

C. COMPARISON BETWEEN THE SOLUTIONS TO THE EXACT EQUATIONS AND THE APPROXIMATE EQUATIONS

The approximate analytical solution is obtained in part I, section I. The initial conditions used in the numerical solution to the exact equations are once again applied to the approximate solution in order to find the values of the integrating constants. A computer program was written to obtain numerical values for the approximate solution and results were printed at the same time intervals as those for the exact equations to be able to compare the two results. Results for $\Delta t = 0.001$ second are included in appendix A.

From the approximate solution results obtained, it was found that these results were not comparable to the results of either of the numerical methods for the exact equations in absolute value. Moreover, the general pattern followed by these results was that the values of ψ decreased slowly and in an oscillating fashion whereas those of α decreased rapidly and uniformly with increase in time.

As the results of the approximate analytical solution do not follow the same pattern as those of either of the numerical methods, and also because neither of the numerical methods has been established to be appropriate for a system of equations such as the present one, a comparison between the approximate solution and the numerical solution is not possible. Another hindrance in determining the suitability of a numerical method for such a problem is the necessity of a very small time interval. Even with as large a step-size as 0.001 second, 1000 readings would be required to obtain the behavior of the mechanical system during the first second of its motion, and this would consume about two minutes of computer time.

D. CONFIRMATION OF RESULTS

In order to confirm the unsuitability of the Runge-Kutta method of order 4, and the Hamming method of order 1, it was tried to change the step-size, and the initial values of the parameters used before. The purpose for doing this was to see if the methods yielded better results when the constant parameters are changed. Several different sets of initial conditions were tried and the results are shown in Tables V through XIV, Appendix A, wherein the readings for ψ and α for different methods are compared. The Runge-Kutta method and the Hamming method were also applied to the approximate equations and the results thereof are included in the above-metnioned tables.

A surprising phenomenon was observed from these tables. It was found that the methods behaved differently with varying values of initial conditions. For instance, in Table V, Appendix A, results from the

2^{nd}, 3^{rd}, and 5^{th} column match well while columns 4 and 6 vary greatly, although these also show an increase in ψ as time increases. In Table VI, the values in columns 2, 3, 4, 5 all agree with each other, whereas the values in column 6 remain unchanged. In Table XI values in all the 5 columns nearly agree with each other. In Table XII, the values given by the numerical methods (columns 3, 4, 5, 6) seem to be close to each other whereas those given by the approximate analytical method (column 2) differs radically from these.

Thus it can be seen that the methods used over here changed their behavior with a change in initial conditions and were in general, unstable. Hence it can be deduced that the numerical methods used in this thesis are not entirely suitable for systems of equations similar to those of the gyrocompass.

E. THE EQUILIBRIUM CONFIGURATION SOUGHT NUMERICALLY

The Runge-Kutta method and the Hamming method were used to determine the equilibrium configuration of the instrument. For the values of the constant parameters used elsewhere in the analysis, it was found that the equilibrium position of the gyrocompass was the one for which,

$$\psi = \dot{\psi} = \dot{\alpha} = 0$$

$$\dot{\phi} = 1000 \text{ radians/second} = \text{constant}$$

$$\alpha = \alpha_o = 0.001786 \text{ radian}$$

The results of the Runge-Kutta method as well as the Hamming method agreed entirely for the equilibrium equation due to the simplicity of the nature of the equation. It was seen that with a change in time there was no change at all in any of the variables except ϕ which indicated a steady rotation of the gyrorotor. The rest of the system was in equilibrium.

Results from the computer program for the equilibrium configuration appear in Table IV, Appendix A.

F. PLOTTING OF GRAPHS

The Runge-Kutta method program for $\Delta t = 0.001$ second was used to plot graphs of different variables. These graphs can be seen in Appendix C.

1. ψ, $\alpha \rightarrow$ Time.

   From the graph it can be seen that α reduces slowly with increase in time, and follows an oscillatory pattern, and ψ increases with increase in time. The motion follows two frequencies - a high frequency and a low frequency, one being superimposed on the other as is observed from the graph.

2. ϕ, $\dot{\phi} \rightarrow$ Time.

   From this graph, ϕ is observed to be following a periodic motion whereas $\dot{\phi}$ slowly reduces with increase in time. Whether this motion of $\dot{\phi}$ is periodic or not, cannot be predicted from the span of time available.

3. $\dot{\alpha}$, $\dot{\psi} \rightarrow$ Time.

   $\dot{\alpha}$ and $\dot{\psi}$ are seen to vary periodically with increase in time.

4. $\psi \rightarrow \alpha$.

The variation of ψ with respect to α is also found to be periodic.

All the graphs have been plotted on a digital computer, using a plotter subroutine.

VI. CONCLUSION

From the discussion of the problem and its solution, the following was concluded, based upon the results obtained from the numerical and the analytical solution.

1. The results of the numerical solution and the approximate analytical solution during the first second of motion of the system do not agree closely with each other.
2. The results of the two numerical methods used, namely, the Runge-Kutta method of order 4 and the Hamming method of order 1 are different for different time intervals, and different initial conditions used. Moreover these results do not compare with each other favourably. However, this conclusion is also based on the readings available for the first second of motion of the system.
3. For a system of equations such as the exact equations of motion of the gyrocompass the Runge-Kutta method of order 4, and the Hamming method of order 1 are not entirely appropriate.
4. The equilibrium configuration for the gyrocompass is one wherein $\psi = 0$, and $\alpha = \alpha_o$. The motion of the gyrorotor is steady in this state, independent of time.

SUGGESTIONS FOR FURTHER WORK

Considering the importance of the gyrocompass as an indispensible tool in inertial navigation a few suggestions for further work are made as follows:

1. To try to find a numerical, multistep method that is suitable for application to "stiff" systems of ordinary differential equations (such as the present one), regardless of the initial conditions used.
2. To try to obtain the solution of the exact equation of motion for a substantial period of time by increasing the step-size or otherwise, retaining the accuracy of the results as well, while doing so.

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VITA

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APPENDICES

APPENDICES

Appendix A: Tables I through XIV

Appendix B: Computer Programs

1. Runge-Kutta Method for Exact Equations
2. Hamming Method for Exact Equations

   (All subroutines according to Program No. 1)
3. Hamming Method for Approximate Equations

   (Subroutines Four, Five, and Six according to Program No. 1)
4. Exact Solution to Approximate Equations
5. Runge-Kutta Method for Equilibrium Configuration

   (All subroutines according to Program No. 1)

Appendix C: Graphs

1. ψ, α $\rightarrow$ Time
2. ϕ, $\dot{\phi}$ $\rightarrow$ Time
3. $\dot{\psi}$, $\dot{\alpha}$ $\rightarrow$ Time
4. $\alpha \rightarrow \psi$

APPENDIX A

TABLE I

COMPARISON OF RESULTS FROM COMPUTER PROGRAMS FOR ψ and α

Time	Runge-Kutta		Hamming		Approximate Method	
	ψ	α	ψ	α	ψ	α
0.000	0.100000	0.100000	0.100000	0.100000	0.100000	0.100000
0.050	0.100116	0.099995	0.100004	0.099954	0.098763	-0.000406
0.100	0.100235	0.099986	0.100007	0.099910	0.097617	-0.101603
0.150	0.100359	0.099976	0.100010	0.099865	0.098901	-0.202252
0.200	0.100487	0.099970	0.100013	0.099820	0.099436	-0.300139
0.250	0.100615	0.099971	0.100016	0.099776	0.097395	-0.398475
0.300	0.100741	0.099976	0.100019	0.099731	0.096253	-0.498690
0.350	0.100862	0.099983	0.100021	0.099686	0.097277	-0.597074
0.400	0.100979	0.099987	0.100024	0.099642	0.096792	-0.692530
0.450	0.101094	0.099985	0.100027	0.099598	0.094122	-0.788680
0.500	0.101211	0.099977	0.100030	0.099553	9.093055	-0.885774

TABLE II

COMPARISON OF RESULTS FROM COMPUTER PROGRAMS FOR $\dot{\psi}$ AND $\dot{\alpha}$

Time	Runge-Kutta		Hamming	
	$\dot{\psi}$	$\dot{\alpha}$	$\dot{\psi}$	$\dot{\alpha}$
0.00	0.000000	0.000000	0.000000	0.000000
0.05	0.000661	-0.002162	0.008704	-0.040194
0.10	0.002285	-0.003166	0.017273	-0.081051
0.15	0.004002	-0.002479	0.025843	-0.121906
0.20	0.004896	-0.000472	0.034413	-0.162756
0.25	0.004492	0.001779	0.042982	-0.203605
0.30	0.003007	0.003072	0.051552	-0.244454
0.35	0.001240	0.002717	0.060122	-0.285303
0.40	0.000134	0.000910	0.068690	-0.326152
0.45	0.000279	-0.001381	0.077258	-0.367001
0.50	0.001595	-0.002928	0.085655	-0.407033

TABLE III

COMPARISON OF RESULTS FROM COMPUTER PROGRAMS FOR ϕ AND $\dot{\phi}$

	Runge-Kutta		Hamming	
Time	ϕ	$\dot{\phi}$	ϕ	$\dot{\phi}$
0.00	0.000000	1000.000000	0.000000	1000.000000
0.05	6.017457	999.993896	5.085742	999.953369
0.10	5.751435	999.987305	4.361202	999.904541
0.15	5.485135	999.980469	3.650125	999.855713
0.20	5.218540	999.974121	2.965577	999.806885
0.25	4.951666	999.968750	1.833585	999.758057
0.30	4.684505	999.964111	1.419412	999.709229
0.35	4.417057	999.959229	0.739189	999.660400
0.40	4.149324	999.954102	0.315885	999.611572
0.45	3.881302	999.948242	5.944942	999.562744
0.50	3.612989	999.941650	4.220536	999.514893

TABLE IV

RESULTS FOR THE EQUILIBRIUM CONFIGURATION

Time	ψ	α	ϕ	$\dot{\psi}$	$\dot{\alpha}$	$\dot{\phi}$
0.00	0.000000	0.001786	0.000000	0.000000	0.000000	1000.000000
0.05	0.000000	0.001786	6.017648	0.000000	0.000000	1000.000000
0.10	0.000000	0.001786	5.752097	0.000000	0.000000	1000.000000
0.15	0.000000	0.001786	5.486560	0.000000	0.000000	1000.000000
0.20	0.000000	0.001786	5.221023	0.000000	0.000000	1000.000000
0.25	0.000000	0.001786	4.955485	0.000000	0.000000	1000.000000
0.30	0.000000	0.001786	4.689947	0.000000	0.000000	1000.000000
0.35	0.000000	0.001786	4.424410	0.000000	0.000000	1000.000000
0.40	0.000000	0.001786	4.158873	0.000000	0.000000	1000.000000
0.45	0.000000	0.001786	3.893335	0.000000	0.000000	1000.000000
0.50	0.000000	0.001786	3.627797	0.000000	0.000000	1000.000000

TABLE V

COMPARISON BETWEEN APPROXIMATE SOLUTIONS AND NUMERICAL SOLUTIONS FOR ψ, SUBJECT TO THE FOLLOWING INITIAL CONDITIONS:

$\psi_o = 0.00$ $\alpha_o = 0.05$ $\phi_o = 0.0$

$\dot{\psi}_o = 0.00$ $\dot{\alpha}_o = 0.00$ $\dot{\phi}_o = 1000.0$

Time in Seconds	Solution to Approximate Equations-	Runge-Kutta Solution to Approximate Equations-	Hamming Solution to Approximate Equations-	Runge-Kutta Solution to Exact Equations-	Hamming Solution to Exact Equations-
0.000	0.000000	0.000000	0.000000	0.000000	0.000000
0.025	0.000030	0.000030	0.000000	0.000034	0.000001
0.050	0.000059	0.000060	0.000000	0.000066	0.000001
0.075	0.000090	0.000091	0.000000	0.000095	0.000002
0.100	0.000119	0.000121	0.000000	0.000122	0.000003
0.125	0.000148	0.000151	0.000000	0.000149	0.000004
0.150	0.000179	0.000181	0.000000	0.000176	0.000004
0.175	0.000208	0.000211	0.000001	0.000206	0.000005
0.200	0.000238	0.000242	0.000001	0.000239	0.000006
0.225	0.000268	0.000272	0.000001	0.000273	0.000007
0.250	0.000296	0.000302	0.000001	0.000306	0.000007
0.275	0.000326	0.000332	0.000001	0.000338	0.000008
0.300	0.000356	0.000362	0.000001	0.000367	0.000009
0.325	0.000384	0.000393	0.000001	0.000394	0.000010
0.350	0.000414	0.000423	0.000001	0.000420	0.000010
0.375	0.000442	0.000453	0.000001	0.000448	0.000011
0.400	0.000471	0.000483	0.000001	0.000479	0.000012
0.425	0.000500	0.000513	0.000001	0.000512	0.000013
0.450	0.000530	0.000544	0.000001	0.000546	0.000013
0.475	0.000557	0.000574	0.000002	0.000579	0.000014
0.500	0.000585	0.000604	0.000002	0.000610	0.000015

TABLE VI

COMPARISON BETWEEN APPROXIMATE SOLUTIONS AND NUMERICAL SOLUTIONS FOR ψ, SUBJECT TO THE FOLLOWING INITIAL CONDITIONS:

$\psi_o = 0.0$ $\alpha_o = 0.0$ $\phi_o = 0.0$
$\dot{\psi}_o = 0.0$ $\dot{\alpha}_o = 0.0$ $\dot{\phi}_o = 1000.0$

Time in Seconds	Solution to Approximate Equations-	Runge-Kutta Solution to Approximate Equations-	Hamming Solution to Approximate Equations-	Runge-Kutta Solution to Exact Equations-	Hamming Solution to Exact Equations-
0.000	0.000000	0.000000	0.000000	0.000000	0.000000
0.025	-0.000001	-0.000001	-0.000001	-0.000001	0.000000
0.050	-0.000002	-0.000002	-0.000002	-0.000002	0.000000
0.075	-0.000003	-0.000003	-0.000003	-0.000003	0.000000
0.100	-0.000004	-0.000004	-0.000004	-0.000005	0.000000
0.125	-0.000005	-0.000006	-0.000005	-0.000005	0.000000
0.150	-0.000007	-0.000007	-0.000006	-0.000007	0.000000
0.175	-0.000008	-0.000008	-0.000007	-0.000008	0.000000
0.200	-0.000009	-0.000009	-0.000007	-0.000009	0.000000
0.225	-0.000010	-0.000010	-0.000008	-0.000010	0.000000
0.250	-0.000011	-0.000011	-0.000009	-0.000011	0.000000
0.275	-0.000012	-0.000012	-0.000010	-0.000012	0.000000
0.300	-0.000013	-0.000013	-0.000011	-0.000013	0.000000
0.325	-0.000014	-0.000015	-0.000012	-0.000015	0.000000
0.350	-0.000015	-0.000016	-0.000013	-0.000016	0.000000
0.375	-0.000016	-0.000017	-0.000014	-0.000017	0.000000
0.400	-0.000017	-0.000018	-0.000015	-0.000018	0.000000
0.425	-0.000019	-0.000019	-0.000016	-0.000019	0.000000
0.450	-0.000020	-0.000020	-0.000017	-0.000020	0.000000
0.475	-0.000021	-0.000021	-0.000018	-0.000021	0.000000
0.500	-0.000022	-0.000022	-0.000019	-0.000023	0.000000

TABLE VII

COMPARISON BETWEEN APPROXIMATE SOLUTIONS AND NUMERICAL SOLUTIONS FOR ψ, SUBJECT TO THE FOLLOWING INITIAL CONDITIONS:

$\psi_o = 0.1$ $\quad$ $\alpha_o = 0.1$ $\quad$ $\phi_o = 0.0$

$\dot{\psi}_o = 0.0$ $\quad$ $\dot{\alpha}_o = 0.0$ $\quad$ $\dot{\phi}_o = 1000.0$

Time in Seconds	Solution to Approximate Equations-	Runge-Kutta Solution to Approximate Equations-	Hamming Solution to Approximate Equations-	Runge-Kutta Solution to Exact Equations-	Hamming Solution to Exact Equations-
0.000	0.100000	0.100000	0.100000	0.100000	0.100000
0.025	0.098038	0.100061	0.100081	0.100058	0.100002
0.050	0.098763	0.100121	0.100155	0.100116	0.100004
0.075	0.099790	0.100183	0.100250	0.100175	0.100005
0.100	0.097617	0.100244	0.100335	0.100235	0.100007
0.125	0.099378	0.100305	0.100420	0.100296	0.100008
0.150	0.098901	0.100366	0.100505	0.100359	0.100010
0.175	0.097422	0.100427	0.100590	0.100422	0.100011
0.200	0.099436	0.100488	0.100675	0.100487	0.100013
0.225	0.097611	0.100549	0.100760	0.100551	0.100014
0.250	0.097396	0.100610	0.100845	0.100615	0.100016
0.275	0.098734	0.100671	0.100930	0.100679	0.100017
0.300	0.096253	0.100732	0.101015	0.100741	0.100019
0.325	0.097295	0.100794	0.101100	0.100802	0.100020
0.350	0.097277	0.100855	0.101185	0.100862	0.100021
0.375	0.095072	0.100916	0.101270	0.100921	0.100023
0.400	0.096792	0.100977	0.101355	0.100978	0.100024
0.425	0.095276	0.101038	0.101440	0.101036	0.100026
0.450	0.094122	0.101099	0.101524	0.101096	0.100027
0.475	0.095612	0.101160	0.101606	0.101152	0.100029
0.500	0.093055	0.101221	0.101691	0.101211	0.100030

TABLE VIII

COMPARISON BETWEEN APPROXIMATE SOLUTIONS AND NUMERICAL SOLUTIONS FOR ψ, SUBJECT TO THE FOLLOWING INITIAL CONDITIONS:

$\psi_o = 0.1$ $\alpha_o = 0.1$ $\phi_o = 0.0$

$\dot{\psi}_o = 0.0$ $\dot{\alpha}_o = 0.0$ $\dot{\phi}_o = 10.0$

Time in Seconds	Solution to Approximate Equations-	Runge-Kutta Solution to Approximate Equations-	Hamming Solution to Approximate Equations-	Runge-Kutta Solution to Exact Equations-	Hamming Solution to Exact Equations-
0.000	0.100000	0.100000	0.100000	0.100000	0.100000
0.025	0.100179	0.100178	0.100000	0.097431	0.099986
0.050	0.101389	0.101388	0.100000	0.071558	0.099974
0.075	0.104461	0.104459	0.100000	-0.061886	0.099962
0.100	0.109857	0.109855	0.100000	-0.641844	0.099950
0.125	0.117586	0.117582	0.100000	-2.104126	0.099938
0.150	0.127196	0.127192	0.100000	-1.921922	0.099926
0.175	0.137867	0.137862	0.100000	-0.860506	0.099914
0.200	0.148571	0.148566	0.100000	-1.967890	0.099902
0.225	0.158274	0.158267	0.100000	-1.926075	0.099890
0.250	0.166137	0.166129	0.100000	-1.900066	0.099878
0.275	0.171683	0.171673	0.100000	-1.959949	0.099866
0.300	0.174889	0.174879	0.100000	-4.889123	0.099854
0.325	0.176194	0.176182	0.100000	-11.496193	0.099843
0.350	0.176408	0.176395	0.100000	8.663445	0.099831
0.375	0.176556	0.176542	0.100000	19.793655	0.099819
0.400	0.177675	0.177660	0.100000	18.260391	0.099807
0.425	0.180613	0.180598	0.100000	31.727722	0.099795
0.450	0.185860	0.185844	0.100000	33.639908	0.099783
0.475	0.193452	0.193436	0.100000	31.952759	0.099771
0.500	0.202965	0.202948	0.100000	14.914494	0.099760

TABLE IX

COMPARISON BETWEEN APPROPRIATE SOLUTIONS AND NUMERICAL SOLUTIONS FOR ψ, SUBJECT TO THE FOLLOWING INITIAL CONDITIONS:

$\psi_o = 0.0$ $\quad \alpha_o = 0.001$ $\quad \phi_o = 0.0$

$\dot{\psi}_o = 0.0$ $\quad \dot{\alpha}_o = 0.0$ $\quad \dot{\phi}_o = 1.0$

Time in Seconds	Solution to Approximate Equations-	Runge-Kutta Solution to Approximate Equations-	Hamming Solution to Approximate Equations-	Runge-Kutta Solution to Exact Equations-	Hamming Solution to Exact Equations-
0.000	0.000000	0.000000	0.000000	0.000000	0.000000
0.025	0.000000	0.000000	0.000000	-0.000026	0.002042
0.050	0.000001	0.000001	0.000000	-0.000219	0.002941
0.075	0.000005	0.000005	0.000000	-0.000784	0.002901
0.100	0.000011	0.000011	0.000000	-0.002023	0.002941
0.125	0.000022	0.000022	0.000000	-0.004406	0.002941
0.150	0.000036	0.000036	0.000000	-0.008685	0.002941
0.175	0.000058	0.000058	0.000000	-0.016092	0.002941
0.200	0.000085	0.000085	0.000000	-0.028634	0.002941
0.225	0.000118	0.000118	0.000000	-0.049606	0.002940
0.250	0.000157	0.000157	0.000000	-0.084397	0.002940
0.275	0.000203	0.000203	0.000000	-0.141799	0.002940
0.300	0.000255	0.000255	0.000000	-0.141799	0.002940
0.300	0.000255	0.000255	0.000000	-0.236018	0.002940
0.325	0.000313	0.000313	0.000000	-0.389348	0.002940
0.350	0.000377	0.000377	0.000000	-0.633955	0.002940
0.375	0.000445	0.000445	0.000000	-1.005001	0.002940
0.400	0.000517	0.000517	0.000000	-1.502783	0.002939
0.425	0.000592	0.000592	0.000000	-2.013656	0.002939
0.450	0.000669	0.000669	0.000000	-2.312171	0.002939
0.475	0.000747	0.000747	0.000000	-2.245726	0.002939
0.500	0.000824	0.000824	0.000000	-1.836017	0.002939

TABLE X

COMPARISON BETWEEN APPROXIMATE SOLUTIONS AND NUMERICAL SOLUTIONS FOR α, SUBJECT TO THE FOLLOWING INITIAL CONDITIONS:

$\psi_o = 0.0$ $\alpha_o = 0.001$ $\phi_o = 0.0$

$\dot{\psi}_o = 0.0$ $\dot{\alpha}_o = 0.0$ $\dot{\phi}_o = 1.0$

Time in Seconds	Solution to Approximate Equations-	Runge-Kutta Solution to Approximate Equations-	Hamming Solution to Approximate Equations-	Runge-Kutta Solution to Exact Equations-	Hamming Solution to Exact Equations-
0.000	0.001000	0.001000	0.001000	0.001000	0.001000
0.025	0.000987	0.000987	0.001000	0.000997	0.000034
0.050	0.000948	0.000948	0.001000	0.000988	0.000033
0.075	0.000885	0.000885	0.001000	0.000971	0.000033
0.100	0.000800	0.000800	0.001000	0.000941	0.000033
0.125	0.000693	0.000693	0.001000	0.000891	0.000033
0.150	0.000570	0.000570	0.001000	0.000808	0.000032
0.175	0.000432	0.000432	0.001000	0.000671	0.000032
0.200	0.000284	0.000284	0.001000	0.000447	0.000032
0.225	0.000130	0.000130	0.001000	0.000079	0.000032
0.250	-0.000025	-0.000025	0.001000	-0.000525	0.000032
0.275	-0.000179	-0.000179	0.001000	-0.001515	0.000031
0.300	-0.000325	-0.000325	0.001000	-0.003136	0.000031
0.325	-0.000461	-0.000461	0.001000	-0.005788	0.000031
0.350	-0.000582	-0.000582	0.001000	-0.010105	0.000031
0.375	-0.000686	-0.000685	0.001000	-0.017040	0.000031
0.400	-0.000768	-0.000768	0.001000	-0.027768	0.000030
0.425	-0.000828	-0.000827	0.001000	-0.042844	0.000030
0.450	-0.000862	-0.000862	0.001000	-0.060708	0.000030
0.475	-0.000871	-0.000871	0.001000	-0.078885	0.000030
0.500	-0.000855	-0.000854	0.001000	-0.097813	0.000029

TABLE XI

COMPARISON BETWEEN APPROXIMATE SOLUTIONS AND NUMERICAL SOLUTIONS FOR α, SUBJECT TO THE FOLLOWING INITIAL CONDITIONS:

$\psi_o = 0.0$ $\alpha_o = 0.05$ $\phi_o = 0.0$

$\dot{\psi}_o = 0.0$ $\dot{\alpha}_o = 0.0$ $\dot{\phi}_o = 1000.0$

Time in Seconds	Solution to Approximate Equations-	Runge-Kutta Solution to Approximate Equations-	Hamming Solution to Approximate Equations-	Runge-Kutta Solution to Exact Equations-	Hamming Solution to Exact Equations-
0.000	0.050000	0.050000	0.050000	0.050000	0.050000
0.025	0.049992	0.049999	0.050000	0.049998	0.049989
0.050	0.049969	0.049999	0.050000	0.049993	0.049978
0.075	0.049932	0.049999	0.049999	0.049987	0.049968
0.100	0.049878	0.049999	0.049999	0.049983	0.049957
0.125	0.049811	0.049999	0.049999	0.049984	0.049947
0.150	0.049729	0.049999	0.049999	0.049988	0.049936
0.175	0.049630	0.049999	0.049999	0.049994	0.049925
0.200	0.049519	0.049999	0.049998	0.049998	0.049915
0.225	0.049390	0.049999	0.049998	0.050000	0.049904
0.250	0.049248	0.049999	0.049998	0.049997	0.049894
0.275	0.049091	0.049999	0.049998	0.049991	0.049883
0.300	0.048918	0.049999	0.049997	0.049986	0.049873
0.325	0.048732	0.049999	0.049997	0.049983	0.049862
0.350	0.048531	0.049999	0.049997	0.049984	0.049851
0.375	0.048314	0.049999	0.049997	0.049989	0.049841
0.400	0.048084	0.049999	0.049997	0.049994	0.049830
0.425	0.047838	0.049999	0.049996	0.049999	0.049819
0.450	0.047568	0.049999	0.049996	0.049999	0.049809
0.475	0.047294	0.049999	0.049996	0.049995	0.049798
0.500	0.047017	0.049999	0.049996	0.049990	0.049788

TABLE XII

COMPARISON BETWEEN APPROXIMATE SOLUTIONS AND NUMERICAL SOLUTIONS FOR α, SUBJECT TO THE FOLLOWING INITIAL CONDITIONS:

$\psi_o = 0.0$ $\alpha_o = 0.0$ $\phi_o = 0.0$

$\dot{\psi}_o = 0.0$ $\dot{\alpha}_o = 0.0$ $\dot{\phi}_o = 1000.0$

Time in Seconds	Solution to Approximate Equations-	Runge-Kutta Solution to Approximate Equations	Hamming Solution to Approximate Equations-	Runge-Kutta Solution to Exact Equations-	Hamming Solution to Exact Equations-
0.000	0.000000	0.000000	0.000000	0.000000	0.000000
0.025	0.000000	0.000000	-0.000001	0.000001	0.000000
0.050	0.000001	0.000000	-0.000001	0.000001	0.000001
0.075	0.000003	0.000000	-0.000002	0.000000	0.000001
0.100	0.000005	0.000000	-0.000003	0.000001	0.000001
0.125	0.000007	0.000000	-0.000003	0.000001	0.000002
0.150	0.000010	0.000000	-0.000004	0.000000	0.000002
0.175	0.000014	0.000000	-0.000005	0.000000	0.000003
0.200	0.000018	0.000000	-0.000006	0.000001	0.000003
0.225	0.000023	0.000000	-0.000006	0.000000	0.000003
0.250	0.000028	0.000000	-0.000007	0.000000	0.000004
0.275	0.000034	0.000000	-0.000008	0.000001	0.000004
0.300	0.000040	0.000000	-0.000009	0.000000	0.000005
0.325	0.000047	0.000000	-0.000009	0.000000	0.000005
0.350	0.000054	0.000000	-0.000010	0.000001	0.000005
0.375	0.000062	0.000000	-0.000011	0.000001	0.000006
0.400	0.000071	0.000000	-0.000011	0.000000	0.000006
0.425	0.000080	0.000000	-0.000012	0.000001	0.000006
0.450	0.000090	0.000000	-0.000013	0.000001	0.000007
0.475	0.000100	0.000000	-0.000014	0.000000	0.000007
0.500	0.000111	0.000000	-0.000014	0.000001	0.000008

TABLE XIII

COMPARISON BETWEEN APPROXIMATE SOLUTIONS AND NUMERICAL SOLUTIONS FOR α, SUBJECT TO THE FOLLOWING INITIAL CONDITIONS:

$\psi_o = 0.1$ $\qquad$ $\alpha_o = 0.1$ $\qquad$ $\phi_o = 0.0$

$\dot{\psi}_o = 0.0$ $\qquad$ $\dot{\alpha}_o = 0.0$ $\qquad$ $\dot{\phi}_o = 1000.0$

Time in Seconds	Solution to Approximate Equations-	Runge-Kutta Solution to Approximate Equations-	Hamming Solution to Approximate Equations-	Runge-Kutta Solution to Exact Equations-	Hamming Solution to Exact Equations-
0.000	0.100000	0.100000	0.100000	0.100000	0.100000
0.025	0.049219	0.099998	0.099971	0.099998	0.099977
0.050	0.002168	0.099997	0.099938	0.099995	0.099954
0.075	-0.050462	0.099995	0.099905	0.099991	0.099932
0.100	-0.099817	0.099994	0.099873	0.099986	0.099910
0.125	-0.148486	0.099993	0.099840	0.099981	0.099887
0.150	-0.200466	0.099991	0.099807	0.099976	0.099865
0.175	-0.248483	0.099990	0.099774	0.099972	0.099843
0.200	-0.298353	0.099988	0.099741	0.099970	0.099820
0.225	-0.349410	0.099987	0.099709	0.099969	0.099798
0.250	-0.396689	0.099985	0.099676	0.099970	0.099776
0.275	-0.447537	0.099984	0.099643	0.099973	0.099753
0.300	-0.496905	0.099982	0.099610	0.099976	0.099731
0.325	-0.544240	0.099981	0.099578	0.099980	0.099709
0.350	-0.595289	0.099979	0.099545	0.099983	0.099686
0.375	-0.642761	0.099978	0.099512	0.099986	0.099664
0.400	-0.690745	0.099976	0.099479	0.099987	0.099642
0.425	-0.740935	0.099975	0.099446	0.099986	0.099619
0.450	-0.786894	0.099973	0.099414	0.099985	0.099597
0.475	-0.835607	0.099972	0.099381	0.099981	0.099574
0.500	-0.883989	0.099970	0.099349	0.099976	0.099553

TABLE XIV

COMPARISON BETWEEN APPROXIMATE SOLUTIONS AND NUMERICAL SOLUTIONS FOR α, SUBJECT TO THE FOLLOWING INITIAL CONDITIONS:

$\psi_o = 0.1$ $\alpha_o = 0.1$ $\phi_o = 0.0$

$\dot{\psi}_o = 0.0$ $\dot{\alpha}_o = 0.0$ $\dot{\phi}_o = 10.0$

Time in Seconds	Solution to Approximate Equations-	Runge-Kutta Solution to Approximate Equations-	Hamming Solution to Approximate Equations-	Runge-Kutta Solution to Exact Equations-	Hamming Solution to Exact Equations-
0.000	0.100000	0.100000	0.100000	0.100000	0.100000
0.025	0.098719	0.098719	0.099976	0.099678	0.099977
0.050	0.095130	0.095129	0.099974	0.097870	0.099954
0.075	0.089935	0.089934	0.099973	0.089958	0.099932
0.100	0.084157	0.084154	0.099971	0.058033	0.099910
0.125	0.078928	0.078925	0.099970	-0.037396	0.099887
0.150	0.075275	0.075273	0.099968	-0.179788	0.099865
0.175	0.073916	0.073914	0.099967	-0.344374	0.099842
0.200	0.075118	0.075115	0.099965	-0.657047	0.099820
0.225	0.078644	0.078641	0.099964	-1.232469	0.099798
0.250	0.083802	0.083799	0.099962	-1.972322	0.099775
0.275	0.089579	0.089575	0.099961	-2.609510	0.099753
0.300	0.094841	0.094837	0.099959	-2.922953	0.099731
0.325	0.098554	0.098549	0.099958	-1.133218	0.099708
0.350	0.099990	0.099984	0.099956	-1.665023	0.099686
0.375	0.098866	0.098859	0.099955	-1.501200	0.099664
0.400	0.095403	0.095395	0.099953	-4.960238	0.099641
0.425	0.090281	0.090272	0.099952	-6.120065	0.099619
0.450	0.084505	0.084496	0.099950	-3.981715	0.099597
0.475	0.079210	0.079201	0.099949	-4.412855	0.099574
0.500	0.075434	0.075426	0.099948	-4.318025	0.099553

APPENDIX B

COMPUTER PROGRAMS

```
            C      STUDY OF THE MOTION OF A GYROCOMPASS WITH SMALL DISTURBANCES,
            C      USING THE RUNGE-KUTTA DIFFERENTIAL EQUATIONS METHOD.
            C
0001              DIMENSION TIMES(1000),PSI(1000),ALPHA(1000),PHI(1000),
                 1SID(1000),ALD(1000),PHD(1000)
            C
0002              COMMON A,C,AO,CO,AI,BI,CI,OC,OS,D12,D34,D56,WL,S1,S2,S12,S22,
                 1C1,C2,OA,OB,OG,OGS,DT,X(6),E(6,500),Y(6),F(4)
                 1,W,EL,EC,R,AI1,BI1,CI1
                 1,SI2DOT,AL2DOT,T1OLD,T2OLD,T1NEW,T2NEW,SIDDOT
            C
            C     W=WEIGHT OF PEND. WT.    EL=LENGTH OF PEND.    EC=ECCENTRICITY
0003              F(1)=0.0
0004              F(2)=0.5
0005              F(3)=0.5
0006              F(4)=1.0
            C
            C     DEFINE PROBLEM PARAMETERS
0007              W=15.0
0008              EL=5.0
0009              EC=0.0
0010              EM=W/386.0
0011              NUM=1000
0012              A=1.8
0013              C=3.0
0014              AO=0.2
0015              CO=0.4
0016              AI=1.15
0017              BI=0.4
0018              CI=1.25
0019              R=3960.0*5280.0*12.0
0020              WL=75.0
0021              OC=7.27*COS(38.0/57.29578)*1.0E-05
0022              OS=7.27*SIN(38.0/57.29578)*1.0E-05
0023              D12=0.0
0024              D34=0.0
0025              D56=0.0
            C
            C     DEFINE INITIAL CONDITIONS
0026              T=0.0
0027              DT=0.001
0028              X(1)=0.1
0029              X(2)=0.1
0030              X(3)=0.0
```

```
      X(4)=0.0
      X(5)=0.0
      X(6)=1000.0
      WRITE(3,77)
   77 FORMAT(T6,'TIME',T22,'PSI',T35,'ALPHA',T52,'PHI',T64,'PSI DOT',
     1T78,'ALPHA DOT',T93,'PHI DOT')
      WRITE(3,100)T,X
C
C     DEFINE RUNGE-KUTTA CONSTANTS
      DO 5 I=1,6
    5 E(I,1)=0.0
      DO 10 K=1,NUM
      L=1
      DO 8 J=1,4
      DO 6 I=1,6
    6 Y(I)=X(I)+E(I,L)*F(J)
      S1=SIN(Y(1))
      S2=SIN(Y(2))
      S12=SIN(2.0*Y(1))/2.0
      S22=SIN(2.0*Y(2))/2.0
      C1=COS(Y(1))
      C2=COS(Y(2))
      CALL OMEGA
      DO 7 I=1,6
    7 CALL CONSTN(I,J)
    8 L=J
      TIMES(K)=K*DT
      DO 9 I=1,6
    9 X(I)=X(I)+(E(I,1)+2.0*(E(I,2)+E(I,3))+E(I,4))/6.0
      IF(X(3)-6.2831853)12,12,11
   11 NPI=X(3)/6.2831853
      FNP=NPI
      X(3)=X(3)-FNP*6.2831853
   12 X(3)=X(3)
      PSI(K)=X(1)
      ALPHA(K)=X(2)
      PHI(K)=X(3)
      SID(K)=X(4)
      ALD(K)=X(5)
      PHD(K)=X(6)
   10 WRITE(3,100)TIMES(K),X
  100 FORMAT(F11.5,6F15.6)
```

```
      CALL PENPOS('KELKAR SUBHASH G',16,1)
      CALL NEWPLT(1.0,1.0,20.0)
      CALL ORIGIN(0.0,0.1000)
      CALL XSCALE(0.0,1.0,18.9)
      CALL YSCALE(0.1,0.1025,9.9)
      CALL XAXIS(0.1)
      CALL YAXIS(0.00025)
      CALL XYPLT(TIMES,PSI,1000,1,-1)
      CALL SYM(6.0,10.0,0.28,'PSI,ALPHA-TIME PLOT',0.0,19)
      CALL NEWPLT(1.0,1.0,20.0)
      CALL ORIGIN(0.0,0.09994)
      CALL XSCALE(0.0,1.0,18.9)
      CALL YSCALE(0.09994,0.1,9.9)
      CALL XYPLT(TIMES,ALPHA,1000,1,-1)
      CALL ENDPLT
      CALL NEWPLT(1.0,1.0,20.0)
      CALL ORIGIN(0.0,0.0)
      CALL XSCALE(0.0,1.0,18.9)
      CALL YSCALE(0.0,8.0,9.9)
      CALL XAXIS(0.1)
      CALL YAXIS(0.8)
      CALL XYPLT(TIMES,PHI,1000,1,-1)
      CALL SYM(6.0,10.0,0.28,'PHI,PHIDOT-TIME PLOT',0.0,20)
      CALL NEWPLT(1.0,1.0,20.0)
      CALL ORIGIN(0.0,999.8)
      CALL XSCALE(0.0,1.0,18.9)
      CALL YSCALE(999.8,1000.0,9.9)
      CALL XYPLT(TIMES,PHD,1000,1,-1)
      CALL ENDPLT
      CALL NEWPLT(1.0,5.5,20.0)
      CALL ORIGIN(0.0,0.0)
      CALL XSCALE(0.0,1.0,18.9)
      CALL YSCALE(0.0,0.005,5.4)
      CALL XAXIS(0.1)
      CALL YAXIS(0.0005)
      CALL XYPLT(TIMES,SID,1000,1,-1)
      CALL SYM(6.0,-4.5,0.28,'VELOCITY PLOT',0.0,13)
      CALL NEWPLT(1.0,5.5,20.0)
      CALL ORIGIN(0.0,0.0)
      CALL XSCALE(0.0,1.0,18.9)
      CALL YSCALE(-0.0035,0.0035,10.8)
      CALL XYPLT(TIMES,ALD,1000,1,-1)
      CALL ENDPLT
```

```
      CALL NEWPLT(1.0,1.0,20.0)
      CALL ORIGIN(0.1,0.09994)
      CALL XSCALE(0.1,0.1025,18.9)
      CALL YSCALE(0.09994,0.1,9.9)
      CALLXAXIS(0.00025)
      CALL YAXIS(0.00001)
      CALL XYPLT(PSI,ALPHA,1000,1,-1)
      CALL SYM(6.0,10.0,0.28,'PSI-ALPHA PLOT',0.0,14)
      CALL ENDPLT
      CALL LSTPLT
      CALL EXIT
      END
```

```
      SUBROUTINE OMEGA
      COMMON A,C,AO,CO,AI,BI,CI,OC,OS,D12,D34,D56,WL,S1,S2,S12,S22,
     1C1,C2,OA,OB,OG,OGS,DT,X(6),E(6,500),Y(6),F(4)
     1,W,EL,FC,R,AI1,BI1,CI1
     1,SI2DOT,AL2DOT,T1OLD,T2OLD,T1NEW,T2NEW,SIDDOT
      OA=-(Y(5)+OC*S1)
      OB=(Y(4)+OS)*C2-OC*C1*S2
      OGS=OC*C1*C2+(Y(4)+OS)*S2
      OG=OGS+Y(6)
      RETURN
      END
```

```
0001          SUBROUTINE CONSTN(I,J)
        C
        C     CONSTN(I,J) IS EMPLOYED TO CALCULATE COEFFICIENTS
0002          COMMON A,C,AO,CO,AI,BI,CI,OC,OS,D12,D34,D56,WL,S1,S2,S12,S22,
             1C1,C2,OA,OB,OG,OGS,DT,X(6),E(6,500),Y(6),F(4)
             1,W,EL,EC,R,AII,BII,CII
             1,SI2DOT,AL2DOT,T1OLD,T2OLD,T1NEW,T2NEW,SIDDOT
0003          IF(I-4)1,2,3
0004        1 E(I,J)=DT*Y(I+3)
0005          RETURN
0006        2 SI2DOT=((A+BI)*(Y(5)*(Y(4)+OS)*S22-OC*Y(4)*S1*S22 +
             1OC*Y(5)*C1*S2*S2 + OB*Y(5)*S2 + OB*OC*S1*S2) + D56*Y(6)*S2 +
             1CI*OC*Y(4)*S1*S22 + CI*OC*Y(5)*C1*S2*S2-CI*Y(5)*(Y(4)+OS)*S22-
             1(C*OG+CI*OGS)*Y(5)*C2-D12*Y(4)+(AO-CO)*OC*OC*S12-(A+AI)*OA*OC*
             1C1-(C*OG+CI*OGS)*OC*S1*C2)/(AO+(A+BI)*C2*C2+CI*S2*S2)
0007          AL2DOT=(-OC*Y(4)*C1)+((-A-BI)*OB*((Y(4)+OS)*S2+OC*C1*C2)+
             1(C*OG+CI*OGS)*((Y(4)+OS)*C2-OC*C1*S2)-WL*S2-D34*Y(5))/(A+AI)
0008          T1OLD=SI2DOT*(AO+(A+BI)*C2*C2+CI*S2*S2)
0009          T2OLD=AL2DOT
0010          T1NEW=(W/386.0)*(R*OC*(EL*Y(5)*C2*C1-EL*Y(4)*S2*S1
             1+EC*Y(4)*C1)-EL*EC*(Y(5)*OA*S2+OC*Y(4)*C1)+R*OC*(EL*OA*C1*S2
             2+EL*OGS*S1-EC*(Y(4)+OS)*C1)+EL*EC*OC*OB*C1+EL*EC*OA*OC*S1*S2)
0011          T2NEW=((W/386.0)*(EL*EC*((Y(4)+OS*Y(5)*S2-OC*Y(4)*S1*S2
             1+OC*Y(5)*C1*C2)-R*OC*EL*Y(5)*S1*S2+R*OC*(-EL*OA*S1*S2+
             2  EL*OC*C1*C1*S2-EL*OS*C1*C2)+EL*EC*OA*((Y(4)+OS)*S2+OC*C1*C2))))
0012          SIDDOT=((A+AI)*(T1OLD+T1NEW)-W*EL*EC*(T2OLD+T2NEW)/386.0)/
             1((A+AI)*(AO+(A+BI)*C2*C2+CI*S2*S2)-(W*EL*EC/386.0)**2*(C2))
0013          CALL FOUR(I,J)
0014          RETURN
0015        3 IF(I-5)5,5,6
0016        5 CALL FIVE(I,J)
0017          RETURN
0018        6 CALL SIX(I,J)
0019          RETURN
0020          END
```

```
      SUBROUTINE FOUR(I,J)
      COMMON A,C,AO,CO,AI,BI,CI,OC,OS,D12,D34,D56,WL,S1,S2,S12,S22,
     1C1,C2,OA,OB,OG,OGS,DT,X(6),E(6,500),Y(6),F(4)
     1,W,EL,EC,R,AI1,BI1,CI1
     1,SI2DOT,AL2DOT,T1OLD,T2OLD,T1NEW,T2NEW,SIDDOT
      E(I,J)=DT*SIDDOT
      RETURN
      END
```

```
      SUBROUTINE FIVE(I,J)
      COMMON A,C,AO,CO,AI,BI,CI,OC,OS,D12,D34,D56,WL,S1,S2,S12,S22,
     1C1,C2,OA,OB,OG,OGS,DT,X(6),E(6,500),Y(6),F(4)
     1,W,EL,EC,R,AI1,BI1,CI1
     1,SI2DOT,AL2DOT,T1OLD,T2OLD,T1NEW,T2NEW,SIDDOT
      ALDDOT=(T2OLD+T2NEW-W*EL*EC*C2*SIDDOT)/(A+AI)
      E(I,J)=DT*ALDDOT
      RETURN
      END
```

```
      SUBROUTINE SIX(I,J)
      COMMON A,C,AO,CO,AI,BI,CI,OC,OS,D12,D34,D56,WL,S1,S2,S12,S22,
     1C1,C2,OA,OB,OG,OGS,DT,X(6),E(6,500),Y(6),F(4)
     1,W,EL,EC,R,AI1,BI1,CI1
     1,SI2DOT,AL2DOT,T1OLD,T2OLD,T1NEW,T2NEW,SIDDOT
      PH2DOT=-SI2DOT*S2-(D56*Y(5))/C-Y(5)*(Y(4)+OS)*C2 +
     1OC*(Y(4)*S1*C2+Y(5)*C1*S2)
      E(I,J)=DT*PH2DOT
      RETURN
      END
```

2

```
C
C        STUDY OF THE MOTION OF A GYROCOMPASS WITH SMALL DISTURBANCES,
C        USING THE RUNGE-KUTTA DIFFERENTIAL EQUATIONS METHOD.
C        AND THE HAMMINGS FORMULAS
C
         DIMENSION PN(6,1),CN(6,1),PN1(6,1),AMN1(6,1),SN1P(6,1)
        1,CN1(6,1),ERROR(6,1)
         COMMON A,C,AO,CO,AI,BI,CI,OC,OS,D12,D34,D56,WL,S1,S2,S12,S22,
        1C1,C2,DA,DB,DG,DGS,DT,X(6),F(6,500),Y(6),F(4)
        1,W,EL,EC,R,AI1,BI1,CI1
        1,SI2DOT,AL2DOT,T1OLD,T2OLD,T1NEW,T2NEW,SIDDOT,Z(6,500),ZP(6,500)
C
C        W=WEIGHT OF PEND. WT.    EL=LENGTH OF PEND.    EC=ECCENTRICITY
         F(1)=0.0
         F(2)=0.5
         F(3)=0.5
         F(4)=1.0
C
C        DEFINE PROBLEM PARAMETERS
         W=15.0
         EL=5.0
         EC=0.0
         NUM=4
         A=1.8
         C=3.0
         AO=0.2
         CO=0.4
         AI=1.15
         BI=0.4
         CI=1.25
         R=3960.0*5280.0*12.0
         WL=75.0
         OC=7.27*COS(38.0/57.29578)*1.0E-05
         OS=7.27*SIN(38.0/57.29578)*1.0E-05
C
C        DEFINE INITIAL CONDITIONS
         T=0.0
         DT=0.001
         X(1)=0.1
         X(2)=0.1
         X(3)=0.0
         X(4)=0.0
         X(5)=0.0
         X(6)=1000.0
```

84

```
      D12=0.1
      D34=0.1
      D56=0.000001
      Z(1,1)=X(1)
      Z(2,1)=X(2)
      Z(3,1)=X(3)
      Z(4,1)=X(4)
      Z(5,1)=X(5)
      Z(6,1)=X(6)
      WRITE(3,77)
   77 FORMAT(T6,'TIME',T22,'PSI',T35,'ALPHA',T52,'PHI',T64,'PSI DOT',
     1T78,'ALPHA DOT',T93,'PHI DOT')
      WRITE(3,300)T,(Z(NN,1),NN=1,6)
  300 FORMAT(F11.4,6F15.6)
C
C     DEFINE RUNGE-KUTTA CONSTANTS
      DO 5 I=1,6
    5 F(I,1)=0.0
   17 DO 8 K=2,4
      L=1
      DO 18 J=1,4
      DO 6 I=1,6
    6 Y(I)=X(I)+F(I,L)*E(J)
      S1=SIN(Y(1))
      S2=SIN(Y(2))
      S12=SIN(2.0*Y(1))/2.0
      S22=SIN(2.0*Y(2))/2.0
      C1=COS(Y(1))
      C2=COS(Y(2))
      CALL OMEGA
      DO 7 I=1,6
    7 CALL CONSTN(I,J,K)
   18 L=J
      NJ=K
      DO 9 I=1,6
    9 Z(I,NJ)=X(I)+(F(I,1)+2.0*(F(I,2)+F(I,3))+F(I,4))/6.0
      T=(NJ-1)*DT
    8 WRITE(3,100)T,(Z(NN,NJ),NN=1,6)
  100 FORMAT(F11.4,6F15.6)
```

```
C
C      4 STARTING VALUES ARE READ FROM THE PRECEEDING PROGRAM
C
C      START ITERATION BY HAMMINGS METHOD
       T=0.003
       J=5
       DO 39 I=1,6
       PN(I,1)=0.0
   39  CN(I,1)=0.0
       DO 133 M=1,4
  133  READ(1,134)T,(Z(N,M),N=1,6),(ZP(N,M),N=1,6)
  134  FORMAT(F12.6/(6F12.6)/(6F12.6))
   19  T=(J-1)*DT
       K=J
       DO 351 I=1,6
  351  PN1(I,1)=Z(I,(J-4))+(4.*DT/3.)*(2.*ZP(I,(J-1))-ZP(I,(J-2))
      1+2.*ZP(I,(J-3)))
       DO 352 I=1,6
  352  AMN1(I,1)=PN1(I,1)-(112./121.)*(PN(I,1)-CN(I,1))
       DO 353 I=1,6
  353  Z(I,K)=AMN1(I,1)
       DO 354 I=1,6
       CALL CONSTN(I,J,K)
  354  SN1P(I,1)=ZP(I,J)
       DO 355 I=1,6
  355  CN1(I,1)=(1./8.)*(9.*Z(I,(J-1))-Z(I,(J-3))+3.*DT*(SN1P(I,1)
      1+2.*ZP(I,(J-1))-ZP(I,(J-2))))
       DO 356 I=1,6
  356  Z(I,J)=CN1(I,1)+(9./121.)*(PN1(I,1)-CN1(I,1))
       DO 35 I=1,6
       ERROR(I,1)=PN1(I,1)-CN1(I,1)
       PN(I,1)=PN1(I,1)
   35  CN(I,1)=CN1(I,1)
       IF(Z(3,J)-6.2831853)13,13,14
   14  NPIH=Z(3,J)/6.2831853
       FNPH=NPIH
       Z(3,J)=Z(3,J)-FNPH*6.2831853
   13  WRITE(3,100)T,(Z(LL,J),LL=1,6)
       J=J+1
       IF(J-500)19,19,49
   49  CALL EXIT
       END
```

```
C        STUDY OF THE MOTION OF A GYROCOMPASS WITH SMALL DISTURBANCES,
C        USING THE RUNGE-KUTTA DIFFERENTIAL EQUATIONS METHOD.
C        AND THE HAMMINGS FORMULAS
C
         DIMENSION PN(6,1),CN(6,1),PN1(6,1),AMN1(6,1),SN1P(6,1)
        1,CN1(6,1),ERROR(6,1)
         COMMON A,C,AO,CO,AI,BI,CI,OC,OS,D12,D34,D56,WL,S1,S2,S12,S22,
        1C1,C2,DT,X(4),E(4,4),Y(4),F(4),ALDDOT
        1,W,EL,EC,R,AI1,BI1,CI1
        1,SI2DOT,AL2DOT,T1OLD,T2OLD,T1NEW,T2NEW,SIDDOT,Z(6,500),ZP(6,500)
        1,PHIDOT,ETAI,XII
C
C        W=WEIGHT OF PEND. WT.    EL=LENGTH OF PEND.    EC=ECCENTRICITY
         F(1)=0.0
         F(2)=0.5
         F(3)=0.5
         F(4)=1.0
C
C        DEFINE PROBLEM PARAMETERS
         W=15.0
         EL=5.0
         EM=W/386.0
         EC=0.0
         ETAI=1.8
         XII=1.8
         NUM=4
         A=1.8
         C=3.0
         AO=0.2
         CO=0.4
         AI=1.15
         BI=0.4
         CI=1.25
         R=3960.0*5280.0*12.0
         WL=75.0
         OC=7.27*COS(38.0/57.29578)*1.0E-05
         OS=7.27*SIN(38.0/57.29578)*1.0E-05
         D12=0.0
         D34=0.0
         D56=0.0
         PHIDOT=10.0
C
C        DEFINE INITIAL CONDITIONS
         T=0.0
```

```
          DT=0.001
          X(1)=0.1
          X(2)=0.1
          X(3)=0.0
          X(4)=0.0
          Z(1,1)=X(1)
          Z(2,1)=X(2)
          Z(3,1)=X(3)
          Z(4,1)=X(4)
          WRITE(3,300)T,(Z(NN,1),NN=1,4)
      300 FORMAT(F11.4,4F15.6)
C         DEFINE RUNGE-KUTTA CONSTANTS
          DO 5 I=1,4
        5 E(I,1)=0.0
       17 DO 8 K=2,4
          L=1
          DO 18 J=1,4
          DO 6 I=1,4
        6 Y(I)=X(I)+E(I,L)*F(J)
          DO 7 I=1,4
        7 CALL CONSTN(I,J,K)
       18 L=J
          DO 9 I=1,4
        9 Z(I,K)=X(I)+(E(I,1)+2.0*(E(I,2)+E(I,3))+E(I,4))/6.0
          T=(K-1)*DT
        8 WRITE (3,100)T,(Z(NN,K),NN=1,4)
      100 FORMAT(F11.5,4F15.6)
C
C         4 STARTING VALUES ARE READ FROM THE PRECEEDING PROGRAM
C
C         START ITERATION BY HAMMINGS METHOD
          J=5
          DO 39 I=1,4
          PN(I,1)=0.0
       39 CN(I,1)=0.0
          DO 133 M=1,4
      133 READ(1,134)T,(Z(N,M),N=1,4),(ZP(N,M),N=1,4)
      134 FORMAT(F12.6/(4F12.6)/(4F12.6))
       19 T=(J-1)*DT
          K=J
          DO 351 I=1,4
      351 PN1(I,1)=Z(I,(J-4))+(4.*DT/3.)*(2.*ZP(I,(J-1))-ZP(I,(J-2))
         1+2.*ZP(I,(J-3)))
```

```
      DO 352 I=1,4
  352 AMN1(I,1)=PN1(I,1)-(112./121.)*(PN(I,1)-CN(I,1))
      DO 353 I=1,4
  353 Z(I,K)=AMN1(I,1)
      DO 354 I=1,4
      CALL CONSTN(I,J,K)
  354 SN1P(I,1)=ZP(I,J)
      DO 355 I=1,4
  355 CN1(I,1)=(1./8.)*(9.*Z(I,(J-1))-Z(I,(J-3))+3.*DT*(SN1P(I,1)
     1+2.*ZP(I,(J-1))-ZP(I,(J-2))))
      DO 356 I=1,4
  356 Z(I,J)=CN1(I,1)+(9./121.)*(PN1(I,1)-CN1(I,1))
   13 WRITE(3,100)T,(Z(LL,J),LL=1,4)
      DO 35 I=1,4
      ERROR(I,1)=PN1(I,1)-CN1(I,1)
      PN(I,1)=PN1(I,1)
   35 CN(I,1)=CN1(I,1)
      J=J+1
      IF(J-500)19,19,49
   49 CALL EXIT
      END
```

```
      SUBROUTINE CONSTN(I,J,K)
C
C     CONSTN(I,J) IS EMPLOYED TO CALCULATE COEFFICIENTS
      COMMON A,C,AO,CO,AI,BI,CI,OC,OS,D12,D34,D56,WL,S1,S2,S12,S22,
     1C1,C2,DT,X(4),E(4,4),Y(4),F(4),ALDDOT
     1,W,EL,EC,R,AI1,BI1,CI1
     1,SI2DOT,AL2DOT,T1OLD,T2OLD,T1NEW,T2NEW,SIDDOT,Z(6,500),ZP(6,500)
     1,PHIDOT,ETAI,XII
      IF(I-3)1,2,3
    1 E(I,J)=DT*Y(I+2)
      ZP(I,K)=Y(I+2)
      RETURN
    2 SIDDOT=-((C*PHIDOT)/ETAI)*Y(4)-((C*PHIDOT*OC)/(ETAI))*Y(1)
      ALDDOT=((C*PHIDOT)/ETAI)*Y(3)-((WL+C*PHIDOT*OC)/XII)*Y(2)
     1 +(C*PHIDOT*OS)/XII
      CALL FOUR(I,J,K)
      RETURN
    3 CALL FIVE(I,J,K)
      RETURN
      END
```

```
C        SOLUTION TO APPROXIMATE DIFF. EQNS. OF MOTION-
C
C        I.C.---- PSI(0)=0.1,ALPHA(0)=0.1,PSIDOT(0)=0.0,ALPHADOT(0)=0.0
C        PHIDOT=100.0
C
         SIZER=0.1
         ALZER=0.1
         SIDZER=0.0
         ALDZER=0.0
         PHIDOT=100.0
         T=0.0
         WL=75.0
         OC=7.27*COS(38.0/57.29578)*1.0E-05
         OS=7.27*SIN(38.0/57.29578)*1.0E-05
         C=3.0
         ETAI=1.0
         XII=1.8
         ALEM1=(C*PHIDOT)/ETAI
         ALEM2=(C*PHIDOT*OC)/ETAI
         ALEM3=(C*PHIDOT)/XII
         ALEM4=(WL+C*PHIDOT*OC)/XII
         ALEM5=(C*PHIDOT*OS)/XII
         Q=ALEM5/ALEM4
         PP=ALEM2+ALEM4+(ALEM1*ALEM3)
         PQ=4.0*ALEM2*ALEM4
         P1=SQRT(0.5*(PP-SQRT((PP)**2.0-PQ)))
         P2=SQRT(0.5*(PP+SQRT((PP)**2.0-PQ)))
         WRITE(3,40)
      40 FORMAT(29X,'P1',10X,'P2')
         WRITE(3,41)P1,P2
      41 FORMAT(21X,F13.6,10X,F13.6)
         XY=((P1*P1)-ALEM4)/(P1*ALEM3)
         XZ=((P2*P2)-ALEM4)/(P2*ALEM3)
         AN2=(-P2*(ALZER-Q)*XZ - SIDZER)/(P1*XY - P2*XZ)
         AN4=(SIDZER + P1*(ALZER-Q)*XY)/(P1*XY - P2*XZ)
         AN1=(SIZER*P2 - ALDZER*XZ)/(P2*XY - P1*XZ)
         AN3=(XY*ALDZER - P1*SIZER)/(P2*XY - P1*XZ)
         WRITE(3,42)
      42 FORMAT(//17X,'AN1',20X,'AN2',20X,'AN3',20X,'AN4')
         WRITE(3,43)AN1,AN2,AN3,AN4
      43 FORMAT(10X,F13.6,10X,F13.6,10X,F13.6,10X,F13.6)
         WRITE(3,50)
      50 FORMAT(//30X,'TIME',10X,'PSI',12X,'ALPHA')
```

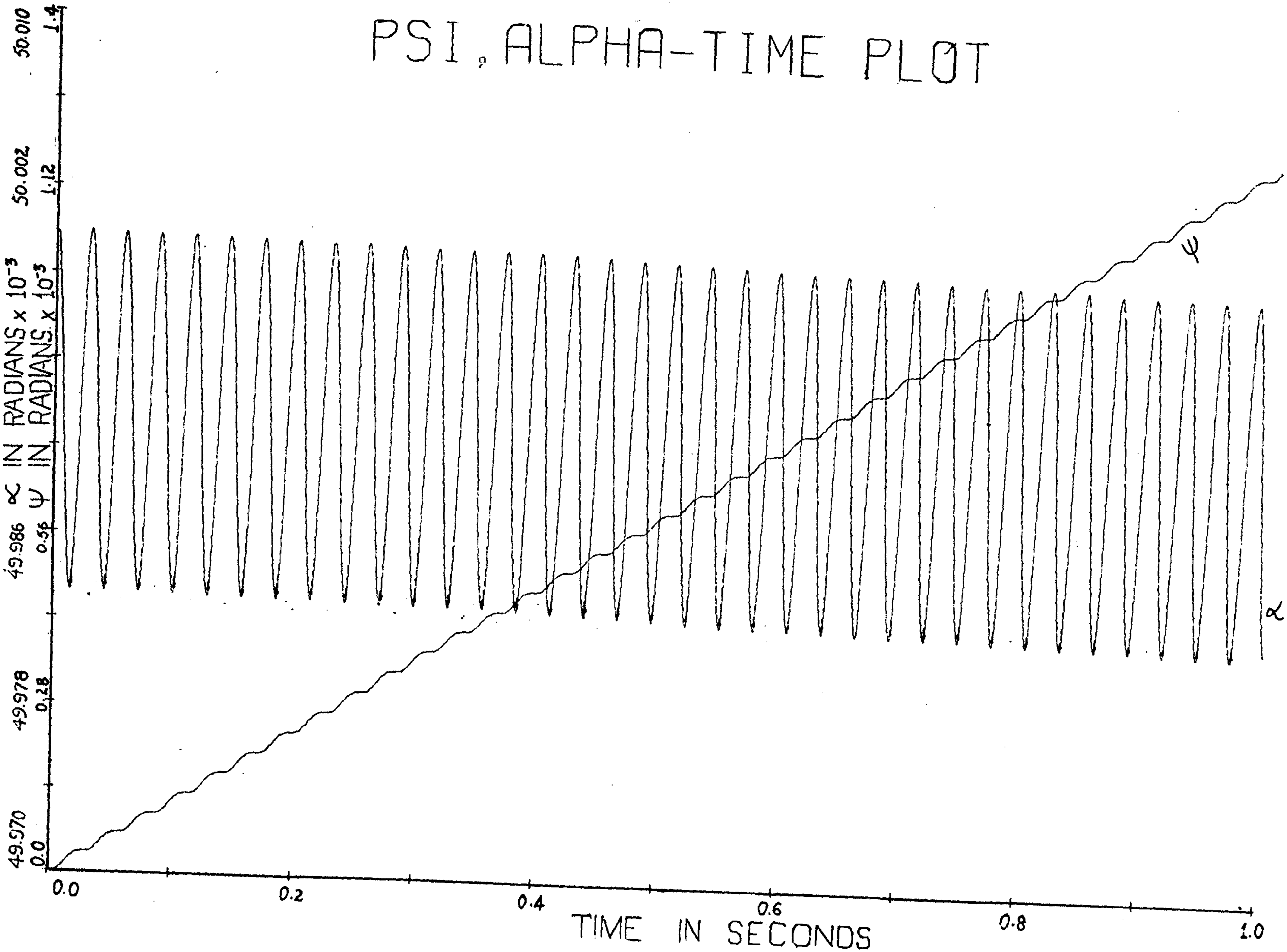
```
       77 PSI=-XY*AN2*SIN(P1*T)+XY*AN1*COS(P1*T)-XZ*AN4*SIN(P2*T)+
         1XZ*AN3*COS(P2*T)
          ALPHA=AN1*SIN(P1*T)+AN2*COS(P1*T)+AN3*SIN(P2*T)+AN4*COS(P2*T)+Q
          SIDOT=-XY*AN2*P1*COS(P1*T) - XY*AN1*P1*SIN(P1*T)-XZ*AN4*P2*COS(P2*
         1 T)-XZ*AN3*P2*SIN(P2*T)
          SI2DOT=XY*AN2*P1*P1*SIN(P1*T) - XY*AN1*P1*P1*COS(P1*T) + XZ*AN4*
         1 P2*P2*SIN(P2*T) - XZ*AN3*P2*P2*COS(P2*T)
          ALDOT=AN1*P1*COS(P1*T) - AN2*P1*SIN(P1*T) + AN3*P2*COS(P2*T)
         1 -AN4*P2*SIN(P2*T)
          AL2DOT=-AN1*P1*P1*SIN(P1*T)-AN2*P1*P1*COS(P1*T)-AN3*P2*P2*SIN
         1 (P2*T) -AN4*P2*P2*COS(P2*T)
          EQ1=SI2DOT + (ALEM1*ALDOT) + (ALEM2*PSI)
          EQ2=AL2DOT - (ALEM3*SIDOT) + (ALEM4*ALPHA) - ALEM5
          WRITE(3,100)T,PSI,ALPHA,EQ1,EQ2
      100 FORMAT(25X,F10.5,4F15.6)
          T=T+0.001
          IF(T-1.0)77,77,700
      700 STOP
          END

/DATA
```

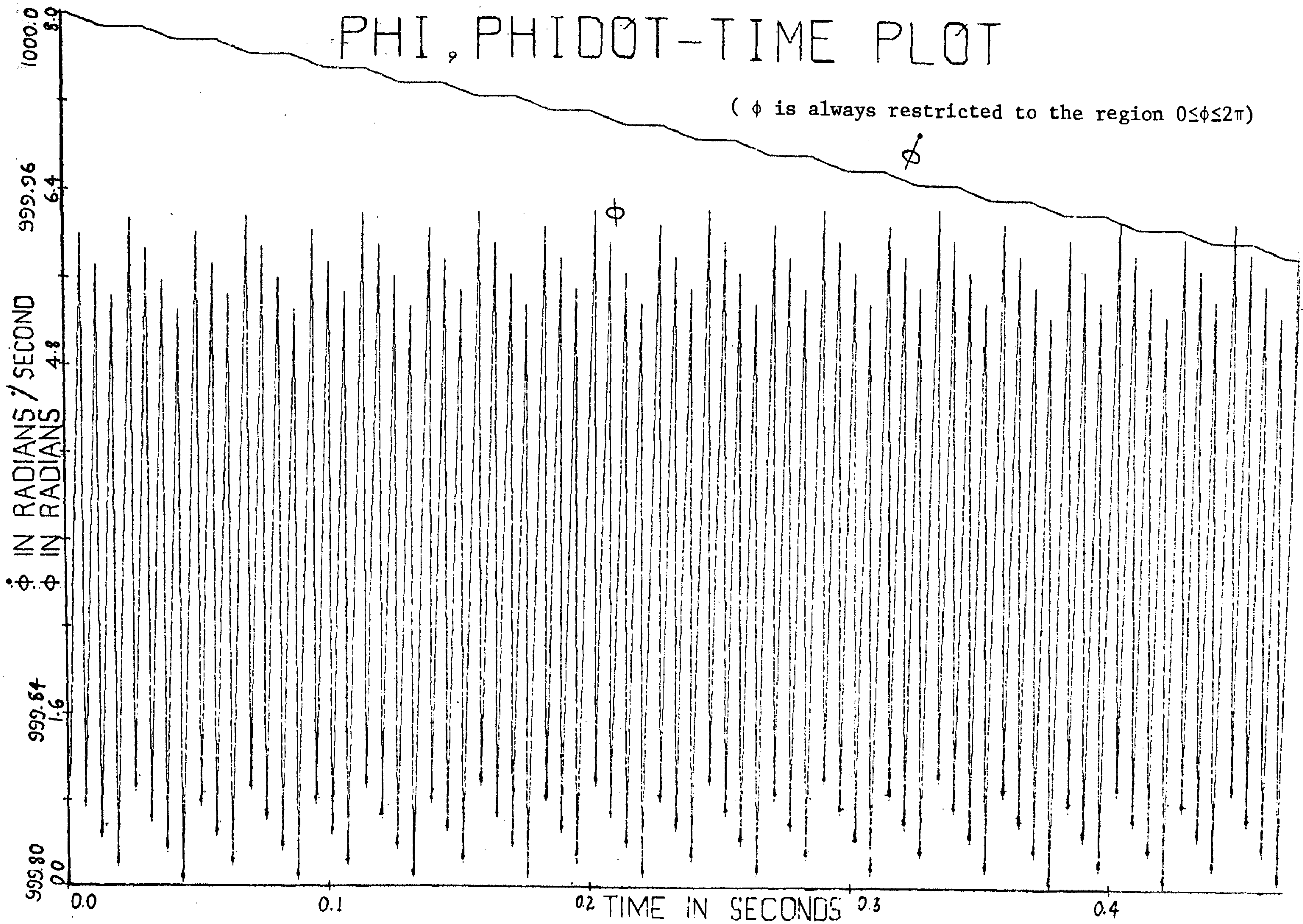
```
C
C     STUDY OF THE MOTION OF A GYROCOMPASS WITH SMALL DISTURBANCES,
C     USING THE RUNGE-KUTTA DIFFERENTIAL EQUATIONS METHOD.
C
      COMMON A,C,AO,CO,AI,BI,CI,OC,OS,D12,D34,D56,WL,S1,S2,S12,S22,
     1C1,C2,DA,DB,DG,DGS,DT,X(6),E(6,500),Y(6),F(4)
     1,W,EL,EC,R,AI1,BI1,CI1
     1,SI2DOT,AL2DOT,T1OLD,T2OLD,T1NEW,T2NEW,SIDDOT,Z(6,500),ZP(6,500)
C
C     W=WEIGHT OF PEND. WT.   EL=LENGTH OF PEND.   EC=ECCENTRICITY
      F(1)=0.0
      F(2)=0.5
      F(3)=0.5
      F(4)=1.0
C
C     DEFINE PROBLEM PARAMETERS
      W=15.0
      EL=5.0
      EC=0.0
      NUM=1000
      A=1.8
      C=3.0
      AO=0.2
      CO=0.4
      AI1=1.15
      AI=AI1+W*EL*EL/386.0
      BI1=0.4
      BI=BI1+W*EC*EC/386.0
      CI1=1.25
      CI=CI1+W*(EL*EL+EC*EC)/386.0
      R=3960.0*5280.0*12.0
      WL=75.0
      OC=7.27*COS(38.0/57.29578)*1.0E-05
      OS=7.27*SIN(38.0/57.29578)*1.0E-05
      D12=0.0
      D34=0.0
      D56=0.0
C
C     DEFINE INITIAL CONDITIONS
      T=0.0
      DT=0.01
      X(1)=0.0
      X(3)=0.0
```

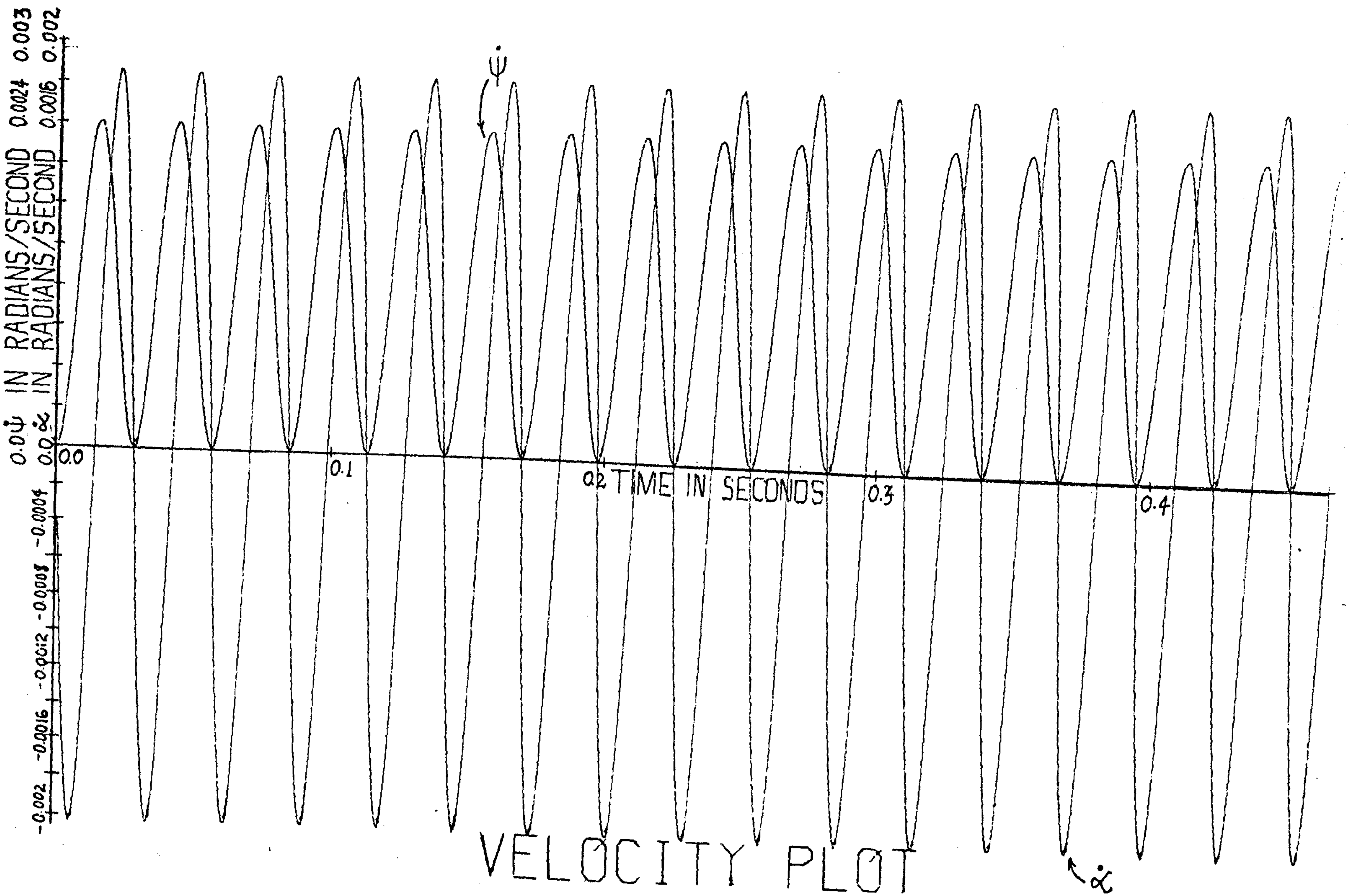
```
      X(4)=0.0
      X(5)=0.0
      X(6)=1000.0
      X(2)=ATAN((C*X(6)*OS)/(W*EL+C*X(6)*OC))
      WRITE(3,77)
   77 FORMAT(T6,'TIME',T22,'PSI',T35,'ALPHA',T52,'PHI',T64,'PSI DOT',
     1T78,'ALPHA DOT',T93,'PHI DOT')
      WRITE(3,100)T,X
C
C     DEFINE RUNGE-KUTTA CONSTANTS
      DO 5 I=1,6
    5 E(I,1)=0.0
      DO 10 K=1,NUM
      L=1
      DO 8 J=1,4
      DO 6 I=1,6
    6 Y(I)=X(I)+E(I,L)*F(J)
      S1=SIN(Y(1))
      S2=SIN(Y(2))
      S12=SIN(2.0*Y(1))/2.0
      S22=SIN(2.0*Y(2))/2.0
      C1=COS(Y(1))
      C2=COS(Y(2))
      CALL OMEGA
      DO 7 I=1,6
    7 CALL CONSTN(I,J)
    8 L=J
      T=T+DT
      DO 9 I=1,6
    9 X(I)=X(I)+(E(I,1)+2.0*(E(I,2)+E(I,3))+E(I,4))/6.0
      IF(X(3)-6.2831853)10,10,11
   11 NPI=X(3)/6.2831853
      FNP=NPI
      X(3)=X(3)-FNP*6.2831853
   10 WRITE(3,100)T,X
  100 FORMAT(F11.4,6F15.6)
      STOP
      END
```

APPENDIX C

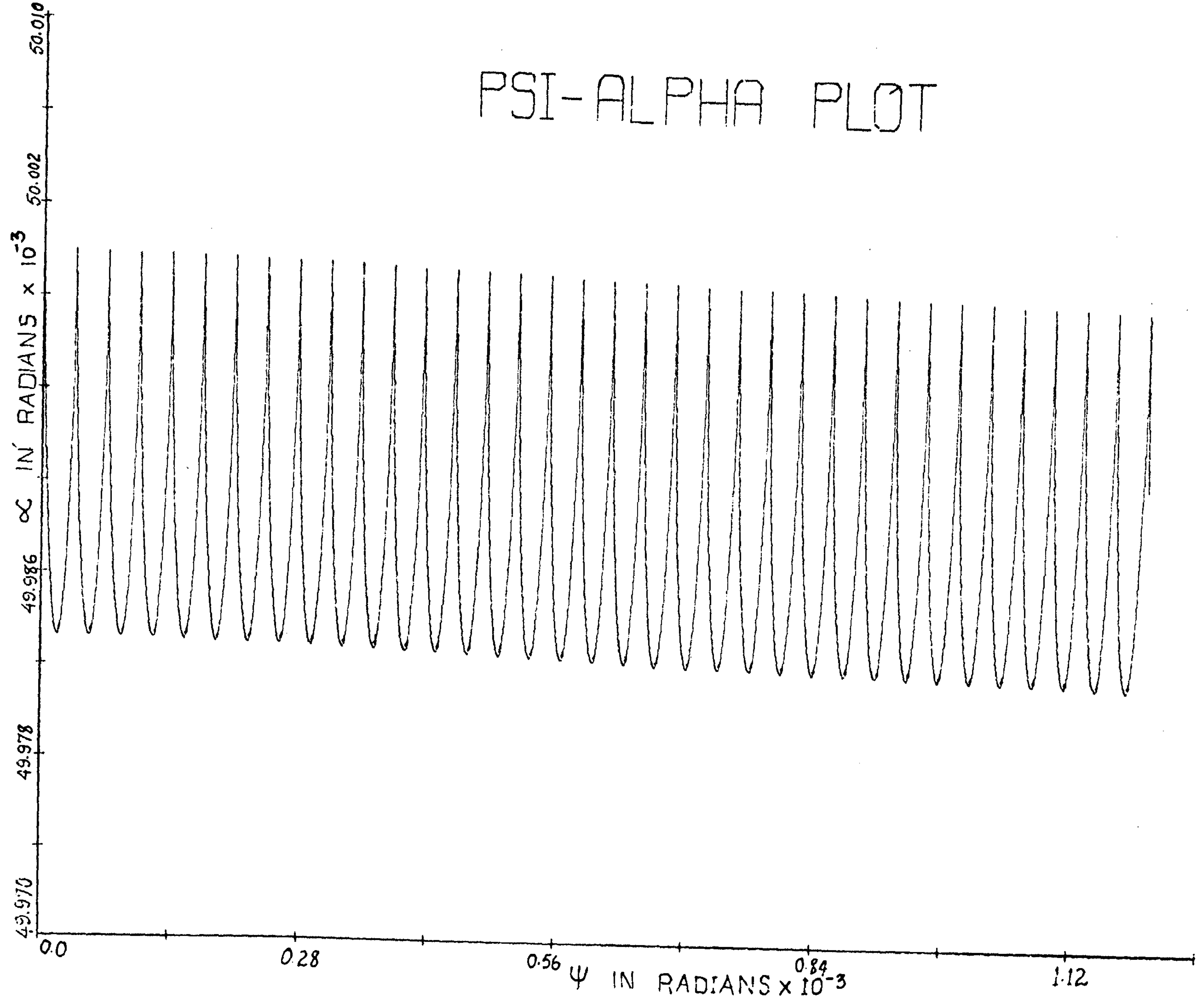
GRAPHS


PSI, ALPHA-TIME PLOT
α IN RADIANS x 10^-3
Ψ IN RADIANS x 10^-3
50.010
1.4
50.002
1.12
49.986
0.56
49.978
0.28
49.970
0.0
0.0
0.2
0.4
0.6
0.8
1.0
TIME IN SECONDS
Ψ
α


PHI, PHIDOT-TIME PLOT
(ϕ is always restricted to the region 0≤ϕ≤2π)
ϕ̇
ϕ
ϕ̇ IN RADIANS / SECOND
ϕ IN RADIANS
1000.0
999.96
999.64
999.80
8.0
6.4
4.8
1.6
0.0
0.0
0.1
0.2
0.3
0.4
TIME IN SECONDS



ψ̇ IN RADIANS/SECOND
α̇ IN RADIANS/SECOND
0.003
0.002
0.0024
0.0016
0.0
-0.0004
-0.0008
-0.0012
-0.0016
-0.002
0.0
0.1
0.2
TIME IN SECONDS
0.3
0.4
ψ̇
α̇
VELOCITY PLOT


PSI-ALPHA PLOT
α IN RADIANS × 10^-3
50.010
50.002
49.986
49.978
49.970
0.0
0.28
0.56
0.84
1.12
Ψ IN RADIANS × 10^-3